FOUR-DIMENSIONAL INTEGRAL EQUATIONS FOR THE MHD DIFFRACTION WAVES IN PLASMA

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Introduction

Let consider low frequency oscillations those can be excited and propagate in plasma. Therefore we shall limit ourselves sufficiently slow development of macroscopic processes. Such a supposition is necessary for the possible application in hydrodynamic description that together with electromagnetic one for the field is expressed by equations of magnetic hydrodynamics [1] relatively to medium velocity \( u(r,t) \), intensity of a magnetic field \( b(r,t) \) and a density of media \( \rho(r,t) \).

In linear magnetic hydrodynamics a wave packet contains seven types of characteristics: two rapid magnetoacoustic waves, two slow magnetoacoustic waves, two Alfven waves and entropy wave. The state vector \( \Psi \) describing the considered wave packet at each spatial point \( M(x, y, z) \) and at any time moment \( t \) is determined for initial and boundary conditions of the above waves.

Under definite physical conditions, for example, at the meeting of the solar wind with the Earth magnetic field, or at sudden inclusion of an electromagnetic field, or during collisions of two gas masses and so on arise strong discontinuities under which not only derivatives of the MHD-values are discontinuous along spatial-temporary coordinates, but these values themselves are also discontinuous ones. Jumps of MHD-values on a surface of discontinuities are determined according to integral laws of conservation or integral balance equations. As to differential equations of magnetohydrodynamics the solutions of those are inaccessible generally, i.e. discontinuity is non-evolutionary and splitted.

The evolution conditions of shock wave easily to find, analyzing the linear boundary conditions, written down in laboratory system of coordinates. In brief these evolutionary conditions can be formulated as follows. Relatively the Alfven disturbances exist two domains of evolution \( u_{\perp 1} > V_1^a \), \( u_{\perp 2} > V_2^a \) (over Alfven) and \( u_{\perp 1} < V_1^a \), \( u_{\perp 2} < V_2^a \) (up to Alfven). Here index 1 means domain ahead of a shock wave, and index 2 behind of it. \( V_1^a \) - the phase Alfven wave velocities accordingly in medium 1. Two evolution domains of shock waves the relation to magnetoacoustic and entropy disturbances: the fast shock wave \( V_1^+ < u_{\perp 1} \), \( V_2^- < u_{\perp 2} < V_2^+ \) and the slow shock wave \( u_{\perp 2} < V_2^- \), \( V_2^- < u_{\perp 1} < V_1^+ \). Here \( V_1^\pm \) - the phase velocities of the fast and slow magnetoacoustic waves. As, \( V_2^- \leq V^a \leq V_2^+ \), then fast shock waves are over alfven, and slow shock waves - up to Alfven .It is necessary to note, that Alfven, tangential and contact types of breaks are always evolutionary.

The second factor leading to the question of wave evolution in magnetic hydrodynamics is connected with taking into account initial conditions. It should be noted that interaction of MHD waves with moving boundary of two media is sufficiently dependent on moving boundary formation (in other terminology - on the surface of MHD-values discontinuity). It can be the interface of motionless and moving media. In this case motion itself is due to external sources. The velocity of
boundary movement can be relative arbitrarily to phase velocities of waves in immovable medium. However, as the boundary and medium move along one site from non-uniformity the difficulties to determine the amount of divergent waves do not arise, can be used the above described principle of evolution of MHD waves.

The discontinuity surface itself can be the boundary formed as a result of propagation of rumpling wave changing medium properties but not actuating the latter for movement. In this case the movement of the medium in both sides is absent, the phase velocity of waves do not depend on the velocity of the boundary movement and the characteristics of a transmitted wave are determined only by parameters of the medium.

Considered above two variants of forming boundary movement reduce to varied non-stationary boundary-value problems resulting in wave scattering patterns of sufficiently different nature. There is a very important value problems resulting in wave scattering patterns of movement reduce to varied non-stationary boundary-value problems. There chain of the integral Volterra equations describe interaction of electromagnetic waves with non-stationarity medium.

For analysis of non-stationarity boundary-value problems in linear magnetic hydrodynamics let us use the integral equations of non-stationarity magnetic hydrodynamics in terms of the constant magnetic field perturbation \( \mathbf{b}(r, t) \), medium perturbations velocity \( \mathbf{u}(r, t) \) and perturbations density \( \rho(r, t) \) [5]:

\[
\mathbf{u}(r, t) = \mathbf{u}_0(r, t) + \hat{\mathbf{K}}_{r,t}^u \mathbf{u}(r, t) + \hat{\mathbf{K}}_{r,t}^b \mathbf{b}(r, t) + \hat{\mathbf{K}}_{r,t}^\rho \rho(r, t) + \hat{\mathbf{K}}_s(u, b),
\]

where \( \mathbf{u}_0(r, t) \) is an incident field and \( \hat{\mathbf{K}}_{r,t}^u, \hat{\mathbf{K}}_{r,t}^b, \hat{\mathbf{K}}_{r,t}^\rho \) are the differential-integral operators of distribution of the velocity, magnetic field and density. Here \( \hat{\mathbf{K}}_{r,t}^u = \hat{\mathbf{K}}_{r,t}^u + \hat{\mathbf{K}}_{r,t}^v, \hat{\mathbf{K}}_{r,t}^v = \hat{\mathbf{K}}_{r,t}^u, \hat{\mathbf{K}}_{r,t}^u = \hat{\mathbf{K}}_{r,t}^u + \hat{\mathbf{K}}_{r,t}^v \) and \( \hat{\mathbf{K}}_{r,t}^b = \hat{\mathbf{K}}_{r,t}^b + \hat{\mathbf{K}}_{r,t}^b \). The operators \( \hat{\mathbf{K}}_{r,t}^u, \hat{\mathbf{K}}_{r,t}^v, \hat{\mathbf{K}}_{r,t}^b \) are responsible for a magnetic tension of MHD media and look like

\[
\hat{\mathbf{K}}_{r,t}^u \mathbf{u} = \hat{\mathbf{G}} \nabla \cdot \mathbf{u},
\]

\[
\hat{\mathbf{K}}_{r,t}^b \mathbf{b} = \hat{\mathbf{G}} \nabla \cdot \mathbf{b},
\]

\[
\hat{\mathbf{K}}_{r,t}^\rho \rho = \hat{\mathbf{G}} \nabla \cdot \rho.
\]

Here, MHD-potentials of velocity, magnetic field and density are defined by analogy to the Hertz potentials in electrodynamics

\[
\begin{bmatrix}
\Pi^u(r, t) \\
\Pi^b(r, t) \\
\Pi^\rho(r, t)
\end{bmatrix} = \int_{V(r)} \mathbf{u}(r', t') \mathbf{b}(r', t') \rho(r', t') dV(r').
\]

And, at last, the operator \( \hat{\mathbf{K}}_s(u, b) \) is defined by presence of surface currents.
The kernel of the integral equation (1) is the Green’s function \( \tilde{G}(r - r', t - t') \) for a free space, determined by parameters \( \{ B_1, V_{A1}, V_{S1}, \rho_1 \} \). Such a representation is typical for the integral equations of scattering problems. \( \tilde{G}(r - r', t - t') \) is characterized by parameters:

\[
\{ e_1, e_2, e_3 \}, e_2 = s_i = B_1 / B_i.
\]

The complete description has been given in [2].

Equation (1) is fully equivalent to the corresponding differential equations of magnetohydrodynamics and the initial and boundary conditions at the surface of inhomogeneous, in laboratory system of coordinates.

Formally, the equation (1) can be considered as a linearised equation of magnetic hydrodynamics with by nonlocal boundary conditions, written in laboratory coordinate system. At the boundary-value problem solution in differential formulation the local boundary conditions can either be satisfied for waves of the same style, or their satisfaction needs usage of several modes. For integral formulation this difficult question is solved automatically, which is caused by physics of the phenomena.

The integral equations (1) contain whole information on scattered waves for the following problem formulation. Assume, that some inhomogeneous, which is characterized by parameters: \( B_2, V_{A2}, V_{S2}, \rho_2, V_m \). occupies a volume \( V(t) \) with time-dependent in a general case boundary. Assume, that considered inhomogeneous is placed and move uniformly with velocity \( U_0 \) in unbounded MHD medium, described by parameters \( B_1, V_{A1}, V_{S1}, \rho_1 (v_m = 0) \) before its excitation by the incident field \( u_0(r, t), b_0(r, t) \) correspondingly.

Let carry out an analysis of ratio (1) in frame of a non-stationary problem of scattering of diffraction on the inhomogeneous of the volume \( V(t) \) using notations and terminology assumed by A.G. Nerukh [4]. So, general ideology is as follows.

Assume, that the object of diffraction before a moment \( t = 0 \) is described by parameters \( B_1, V_{A1}, V_{S1} \) and \( \rho_1 \). At temporary moment \( t = 0 \) the state of an object is changed for \( B_2, V_{A2}, V_{S2}, \rho_2, V_m \) and \( U_0 \). Change of the state results in that, (1) disintegrates in the chain of evolutionally connected expressions.

There are three temporary intervals for the internal field. At \( t < 0 \) the four-dimensional interval of integration of equation (1) for temporary- spatial coordinates is unlimited as the integration is made for the crossing of the region of a transmitted light cone with the top at a point \((t, r)\) and into region given by

\[
\chi(t, r) = \begin{cases} 1, r \in V(t) \\ 0, r \notin V(t) \end{cases}
\]

the characteristic function.

After a zero temporary moment the part of the region of integration limited by hyperplane appears \( t' = 0 \). How the object is in a new state. And two regions of integration appear. One of them is completely in four-dimensional spatial region of events corresponding to the object of diffraction. There is no effect of boundaries of a diffraction object and one is taking into account only change of medium properties in pure appearance. If the equation (1) is used for all the four-dimensional space intervals then it will describe the field in unlimited medium with similar properties as the medium within the object.

Starting from the moment \( t = 1 / \mu d(r) \), where \( d(r) \) is a minimal distance from a point of \( r \) to the boundary of volume \( V(0) \), a transmitted light cone will already cross the boundary of four-dimensional domain \( \chi(t, r) = 1 \) and a boundary of \( V(t) \) will effect on formation of the internal field.

As a result we obtain the following evolution of process of the field interaction with a object of diffraction.

A point of observation is within the region \( V(t) \). 1) Before the zero temporary moment the incident field \( u_0(r, t), b_0(r, t) \) generates the internal fields \( u_1(r, t), b_1(r, t) \). 2) After changing of state of an object at zero temporary there is no effect of the object boundary of an object in the region and incident field does not obviously take part in creation of the internal field, that is the field \( u_2(r, t), b_2(r, t) \) is directly generated by the field of \( u_1(r, t), b_1(r, t) \), formed within the object before changing of the state and conditioned by the field of \( u_0(r, t), b_0(r, t) \) that is the memory of medium. 3) To form the field \( u_3(r, t), b_3(r, t) \) together with the field \( u_1(r, t), b_1(r, t) \) the incident field \( u_0(r, t), b_0(r, t) \) crossing the boundary and field \( u_2(r, t), b_2(r, t) \) take part in the process above mentioned.

A point of observation is beyond the region of \( V(t) \). Then from the expression (1) we receive the square formula. In this case two temporary intervals are allocated. 1) At \( t < 1 / \mu l(r) \), where \( l(r) \) – distance from a observation point at moment \( t=0 \) up to the nearest point of domain \( V(t) \), the external field defined by nondisturbance condition of diffraction object by founding field \( u_1(r, t), b_1(r, t) \). 2) At \( t \geq 1 / \mu l(r) \), an external field, on which already influence and the new condition, will be defined in view of a field \( u_1(r, t), b_1(r, t) \).

In this connection the following algorithm to solve the problem is considered. Firstly the solution of non-disturbance problem that can be considered as steady-
state one with corresponding choice of point parameters is sought. Secondary is sought the solution of the disturbance problem, but without the effect of boundaries of a diffraction object, that is only the disturbance of the medium itself is present. At the third stage the problem is solved with taking into account the boundary-value effects.

If the non-stationarity dissipation problem of MHD waves on the elementary plane boundary it is possible to consider theoretically stage by stage investigated, in sense mathematical realization there is the set of difficulties. First of all transformation of waves an each other on the boundary of interface a much complicates the intermediate mathematical calculations one can note the sufficiently new results of magnetic hydrodynamics arising in the solution of the non-stationary boundary-value problem for Alfven and in common for fast and slow magnetoacoustic waves. Here it is possible to look after complete transformation of MHD waves, as was made. That is wave as though incident on the MHD inhomogeneous and reflected medium having parameters \( B_i, V_{A_i}, V_{S_i}, \rho_i, i = 1,2 \). Let internal medium begins a uniform moment with velocity \( U_0 \) to perpendicularly its boundary in moment \( t = 0 \). Prior to the beginning a movement an inhomogeneous occupied domain \( z > 0 \). Then the law of boundary movement will be set by the formula \( z(t) = U_0 \theta(t) \). Here \( \theta(t) \) is the Heaviside function.

For such a setting a diffraction problem in case of consideration only the Alfven wave is of scalar form. Then integral relationship (1) to find the Alfven field is sought the solution of the propagation of small perturbations in strongly inhomogeneous MHD media.

Let the plane Alfven wave falls onto a plane boundary \( z = 0 \) of two nondissipation media we have \( B_i, V_{Ai}, \rho_i, i = 1,2 \). In short a problem of scattering of Alfven wave by plasma half-space (MHD-inhomogeneity) after a initiating its movement is considered.

The model in which one type of MHD waves is coupling with another through a plane boundary between two media is an important first approximation for the description of the propagation of small perturbations in strongly inhomogeneous MHD media.

For the analysis of the internal field after initial
movement \((t > 0)\) it follows that the field in the region of \(V_{A1} t < z\) does not have either frequency and wave number and the amplitude existed before starting of movement of the Alfven wave is changed and the constant component due to jump of velocity of medium appears. It is connected with that boundary-value effects for the moving boundary do not influence on this field. In the region of \(U_0 t \leq z \leq V_{A1} t\) it is necessary to consider separately cases of «run-away» \(U_0 > 0\) and "encouter" \(U_0 < 0\) movement with for each case we obtain the Volterra two-dimensional equation of the second type with the Fredholm kernel for that was developed sufficiently well the approach fort solution of problem as uniformly convergent Neiman series.

The reflected field is easily restored by means of the quadrature formulas (5) for the already known internal field. Thus it should be noted the sufficient difference of a spectrum of the scattering field in case of the finite temporary moment from the spectrum of the scattered field for the adiabatic inclusion at infinity. The spectrum of this field consists of waves with different frequencies propagating as from the medium boundary as towards boundary itself then as in case of inclusion at infinity there is only the wave reflected from the boundary in transmitted field.

## Conclusions

The superficial analysis of the boundary-value non-stationary problem for Alfven wave has shown the principal possibility of using the method of evolutionary integral equations of non-stationary macroscopic electrodynamical in a case of MHD description of waves in plasma. With the importance of strict mathematical solutions obtained for simple model problems that is the diffraction of one separately taken Alfven wave is that it can be the basis for construction of the approximate solutions of more complex boundary-value problems.

## References