1. INTRODUCTION

It is known that in vacuum the oscillator effectively radiate high numbers of harmonics only in the case that it has a large energy. So, for synchrotron radiation the maximum of radiation is obtained at harmonics with number \( n \sim \frac{\gamma}{\epsilon} \) [1]. Many authors (see, for example, [2-6] and the literature therein) studied the radiation of relativistic particles in periodically inhomogeneous medium. The interest to such radiation is conditioned by that due to Doppler effect there is a possibility to excite effectively the short-wave radiation \( \lambda \sim d / \gamma^2 \).

The short-wave radiation can be exited not only relativistic particles. It is possible one more mechanism of generating of short-wave radiation by nonrelativistic oscillators — excitation of high number harmonics. For this purpose it is necessary, that the motion of nonrelativistic oscillators occurred in periodic inhomogeneous medium.

This mechanism can be utilized for collectively induced excitation of X-radiation in crystals. Really, to create an ensemble of nonrelativistic oscillators it is easier, than an ensemble relativistic. Such ensemble can be produced with the help of external radiation, which one transforms electrons of crystals into oscillators. Density of such oscillators is limited only to density of a solid. Than more the density of individual radiators, the more effectively and faster develops process of induced radiation. Keeping in mind peculiarities of the radiation mechanism which we are investigated, (see below and articles [7-10]), it is possible to expect on creation of sources of an intensive coherent X-radiation with a wavelength \( \lambda \leq d \).

In the given paper theoretical and experimental investigations of excitation of harmonics by ensembles of oscillators — charged particles driving by an external periodic in time electrical field, which moves in the field of an external periodic in space potential or in periodically inhomogeneous medium are carried out. Analytical and numerical analysis of a full self-consistent set of equations are performed. The analytical results are in the good agreement with results of the numerical analysis and qualitatively well be agreed the data of experiment (see, for example [11]).

2. RADIATION OF A PARTICLE MOVING IN PERIODIC POTENTIAL

In the most of nonlinear Hamilton systems, which describe the dynamics of particles, it is possible to sort regions of phase space where trajectories have regular character and where they are stochastic. We are interested with those particles, which are moving regularly, only they can irradiate intensive collective coherent radiation. Therefore in further we shall orient, first of all, by particles with regular dynamics. The influences of stochastic particles at this stage of analysis we'll be neglect, although we will make estimate their number and minimize it.

Let charged particle moves in external periodic in time electrical field \( \vec{E}(t) = \vec{E}_{ext} \cdot \cos(\omega_{ext} \cdot t) \) and in the field of periodic potential with amplitude \( U(z) = U_0 + g \cdot \cos(\kappa \cdot z) \).

For simplicity we’ll consider that motion occur only along z-axis. Then, introduce dimensionless coordinates, we’ll get equations of motion of particles in these fields

\[
\begin{align*}
\dot{p} + \Omega_0^2 \cdot \sin(\xi) &= -e \cdot \cos(\Omega \cdot \tau) \\
\dot{\xi} &= p / \sqrt{1 + p^2} = \beta
\end{align*}
\]

where: \( p = p_z / mc \), \( \xi = \kappa \cdot z \), \( \tau = \kappa \cdot t \), \( \Omega_0^2 = e^2 g / mc^2 \), \( e = eE_{ext}d / 2\pi mc^2 \), \( \beta = V_z / \epsilon \), \( \Omega = \lambda_{ext} / d \), \( \kappa = 2\pi / d \).

At sufficiently small intensities of these fields it is possible to consider nonrelativistic motion of particles in such fields.

\[
\ddot{\xi} + \Omega_0^2 \cdot \sin(\xi) = -e \cdot \cos(\Omega \cdot \tau) \tag{2}
\]

While passing into moving coordinate frame \( \xi = \dot{\xi} + e \cdot \Omega_ \cdot \cos(\Omega \cdot \tau) \) equation (2) takes form...
\[ \zeta = \Omega_0^2 \sum_{n=-\infty}^{\infty} J_n(\mu) \cdot \sin \left( \zeta - (n \cdot \Omega_0^{-1}) \tau + n \cdot \frac{\pi}{2} \right) \]  

where \( J_n(\mu) \) is the Bessel function, \( \mu = \epsilon \cdot \Omega_0^{-1} \).

Equation (3) describe changing of “particle” phase \( \zeta \), at which many of waves acts on. Amplitudes of those waves \( \Omega_0^2 \cdot J_n(\mu) \) are increasing with growing of harmonic’s number and in region \( n \sim \mu \) have a local maximum. Amplitudes of harmonics with number \( n > \mu \) decrease exponentially [12].

\[ J_n(\mu) \sim (2/n)^{1/3} Ait(z) \sim (2/n)^{1/3} (1/2\sqrt{\pi z}) e^{-z} \]  

where: \( z = (2/n)^{1/3} (n-\mu) \), \( \zeta = (2/3)^{3/2} \).

Thus, it is possible to expect, that at motion of the particles in external periodic in time electrical field and in field of the periodic potential, radiation field will contain harmonics with frequencies up to \( n \omega_{ext} \).

Really, considering \( \epsilon >> \Omega_0^2 \) \( (E >> \gamma \cdot k) \), the solution of the equation (2) can be found by a method successive approximations:

\[
\zeta = -\Omega_0 \sin \Omega_0^{-1} \tau \\
-2\Omega_0^2 \sum_{n=0}^{\infty} \frac{J_{2n+1}(\mu)}{(2n+1)} \sin \left( (2n+1) \Omega_0^{-1} \tau \right).
\]

\[
\zeta = \epsilon \Omega_0^2 \cos \Omega_0^{-1} \tau \\
-2\Omega_0^2 \sum_{n=0}^{\infty} \frac{J_{2n+1}(\mu)}{(2n+1)^2} \cos \left( (2n+1) \Omega_0^{-1} \tau \right).
\]

From this expression it is visible, that the velocity of particles and its coordinates are contained odd harmonics of the external electrical field.

Radiation intensity into space angle unit \( d\omega \) with frequency \( \omega = n \omega_{ext} \) is equal to [13]:

\[ dJ_n = \frac{\epsilon}{2\pi} \left| \tilde{A}_n \right|^2 \frac{R_0^2}{c} d\omega , \]

where \( \tilde{H}_\omega = \left[ \hat{k} \hat{A}_\omega \right] \), and Fourier component of vector potential is defined by

\[ \tilde{A}_\omega = \epsilon \left( \frac{\exp(\imath k R_0)}{c \Omega_0^2} \right) \frac{1}{T} \int_0^T \left[ \hat{v}(t) \exp \left[ \imath \left( \omega_{ext} T - \hat{r}(t) \right) \right] \right] dt \, , \]

where \( T = 2\pi / \omega_{ext} \), \( \hat{r}(t) \), \( \hat{v}(t) \) are particle radius-vector and velocity, \( \hat{k} \) is the wave vector, \( R_0 \) is the distance to point of observation. It is easy to see that in our case of one-dimensional motion the radiation has dipole nature.

3. RADIATION SPECTRUM OF PARTICLE AT MOTION IN PERIODIC POTENTIAL

In the common case, it isn't seemed possible to get the analytical dependencies of spectral density from parameters of external fields.

The investigation of spectral characteristic of fields radiated by charged particle moving in external electric field and in field of potential, was carried out by the numerical solution of equations (2) and substitution of its solutions into (7), (6).

For a case of nonrelativistic motion the amplitude of an external electrical field is equal to \( E_{ext} = 10^7 \) V/cm. Frequency of the external electromagnetic field was fixed.

Investigation was carried out for two value of potential period \( d = 0.0025 \lambda_{ext} \) and \( d = 0.00125 \lambda \). Value of potential amplitude varied within \( g = (0 \cdot 0.125) E_{ext} \cdot k^{-1} \).

Initial conditions for particles were equal to \( \zeta(\tau = 0) = \zeta_0 \), \( \dot{\zeta}(\tau = 0) = 0 \). For that the right-hand of (2) had been presented as: \( \epsilon \cdot \cos \left( \Omega_0^{-1} \tau \right) \).

Calculation accuracy was controlled with the help of integral of motion.

\[ I_n = \frac{\rho^2}{2} \left[ \Omega_0^2 \left( \cos(\zeta) - \cos(\zeta_0) \right) + \beta_0^2 \beta(\tau) \cos(\Omega_0^{-1} \tau) \right] \]

its absolute values was less than \( |\rho| < 10^{-30} \).

In absence of the periodic potential influence \( \Omega_0^2 = 0 \) the equation (3) has a simple analytical solution. Motion of the particle is periodic with the frequency \( \omega = \omega_{ext} \), and spectrum of its speed and spectrum of the radiation field are linear.

Presence of space-periodic potential with amplitude of \( \Omega_0^2 = 0.025 \epsilon \) \( (d = 0.0025 \cdot \lambda_{ext}) \) (Fig. 1) give rise to originating multifrequency motion. Passing to variable \( \beta_p(\tau) = \beta(\tau) - \epsilon \Omega_0 \sin(\tau) \) it is visible, that under the action of the potential the particle performs high-frequency oscillations (Fig. 1c), which results in the appearance of external field harmonics both in spectrum of speed, and in spectrum of power of radiated field (Fig. 1d).

**Fig. 1.a. Phase space**

**Fig. 1.b. Spectrum of velocity**

**Fig. 1.c. Influence of potential on particle trajectory**

**Fig. 1.d. Spectrum of radiated field**

Relative maximum fall at harmonic with number \( n = 11 \), that in order of magnitude in is very good accord as position of relative maximum \( n \sim \mu = \epsilon \cdot \Omega_0^{-2} \approx 12 \), and with condition of radiation \( \lambda = d / \beta \).

With growing of amplitude of periodic potential \( \Omega_0^2 = 0.075 \cdot \epsilon \) \( (d = 0.0025 \cdot \lambda_{ext}) \) (Fig. 2) occurs the growing of harmonics amplitudes in spectrum of velocity, and its enriching at intermediate frequencies. Amplitudes of high-frequency oscillations grow up at the influence of periodic potential. Amplitudes of all harmonics in spectrum of radiated field are growing up too. Kind of the spectrum practically has no changes.
For potential amplitude \( \Omega^2 = 0.125 \cdot \varepsilon \) (\( d = 0.0025 \cdot \lambda_{\text{ext}} \)) (Fig. 3) occur qualitative changes of phase plane – the particle motion isn’t localized in limited region of space and represented series of oscillations near locally stabled state. Spectrum of particle velocity and, consequently, spectrum of radiated field lot enriching at all of intermediate frequencies.

Further growing of potential amplitude leads to appearance of non-regular motion of particle hence the spectrum of radiated field also becomes non-regular.

For a potential with period \( d = 1.25 \cdot 10^{-3} \cdot \lambda_{\text{ext}} \) the number of harmonics, both in the spectrum of velocity and radiated field, are proportional to parameter \( \mu \).

At amplitudes of potential \( \Omega^2 = 0.07 \cdot \varepsilon \) (Fig. 4) the motion of particle is quasiregular. Spectrum of velocity and spectrum of radiated field have line structure. The local maximum of spectrum fall on harmonics with number \( n_{\text{max}} = 23 \), that is in very good accord as position of relative maximum \( n \sim \mu = \varepsilon \cdot \Omega^2 \approx 25 \), and with condition of radiation \( \lambda = d / \beta \).

4. SPECTRUM OF OSCILLATOR RADIATION AT MOTION IN PERIODICALLY INHOMOGENEOUS MEDIUM

Let us consider the problem of the radiation of an oscillator in a periodically heterogeneous medium. We will describe heterogeneous medium, where the oscillator moves, by the permeability in the form:

\[
\varepsilon = \varepsilon_0 + \sum_{i=1}^{n} \varepsilon_i \cos(k_i \cdot \hat{r})
\]

where \( q_i \ll 1 \), \( k_i \) is the vector of the reciprocal lattice of the heterogeneous medium. We shall restrict below by the case \( n = 1 \), when the electromagnetic field effectively interacts with one family of crystallographic planes only, i.e. when only two wave diffraction is possible. A charged particle moves in this medium. The trajectory of the particle can be described by the expression:

\[
\hat{r} = \hat{V}_0 + \hat{r}_0 \sin \Omega t
\]

For non-relativistic particles from the conservation lows it follows that \( \omega = \kappa \hat{V} \), \( \kappa >> k \), \( \kappa = k / \beta \), i.e. radiation is like a ‘long-wave’. Consider the case of \( \kappa || \nu || z \), where \( \nu \) is the oscillators velocity. Using Maxwell equations we find for radiation losses of nonrelativistic oscillator [7]

\[
\frac{\partial W}{\partial t} = \left( \frac{\varepsilon^2 \Omega^2 \cdot \beta^2}{3c} \right) \sum_{n=1}^{\infty} \frac{n^4}{m^2} J^2_n(m) \left( \sin \theta \right)^3 d\theta
\]

where \( \beta_\bot = \hat{r}_0 / \varepsilon \) is the ratio of oscillating velocity to velocity of light, \( \hat{k}_\bot \hat{r}_0 = \pm m \) (\( m >> 1 \)); \( \hat{k}_\bot = \hat{k} \pm \hat{\kappa} \). The dependence of radiation power from harmonic number \( n \) is most significant. It is easy to understand that in this case of periodic medium efficiency of radiation increase with grow of the harmonic number. However, for radiation losses of oscillator moving in vacuum (see, for example, [1])

\[
\frac{\partial W}{\partial t} = \frac{\varepsilon^2 \Omega^2}{c} \sum_{n=1}^{\infty} \frac{\pi}{n^2} J^2_n(\theta) \sin \theta \tan^2 \theta d\theta
\]

this effect is impossible because \( \beta_\bot \cos(\theta) \leq 1 \). But as it follows from equation (10), it is very easy to satisfy conditions necessary for radiation of oscillator in the presence of medium.

5. RADIATION OF FLOW OF OSCILLATORS

The fullest description of self-consistent process of interaction of charged particles with an exciting field implies the simultaneous solution of Maxwell equations for the electromagnetic field and equations of charged particles’ motion in exited fields

\[
\frac{\partial \vec{B}}{\partial t} = -c \text{rot} \vec{E},
\]

\[
\frac{d \vec{p}}{dt} = e \vec{E} + \frac{e}{c} \vec{v} \times \vec{B} + e \vec{E}_{\text{ext}} \sin \theta_{\text{ext}} - e \vec{\nabla} U,
\]
It is known, that in periodic medium solution for exited wave is convenient to present in such form:

$$
\vec{E} = \text{Re} \sum \vec{E}_l(t) \exp(ikx + il\kappa z).
$$

(13)

Substitute expressions for fields (13) in the set of equations (12). Averaging the obtained equations on a space phase of disturbance, we’ll obtain the set of equations for finding fields and characteristics of oscillators:

$$
\frac{dp_x}{d\tau} = \text{Re} \sum_l \epsilon_{x,l} \exp(ikx + il\kappa z),
$$

$$
\frac{dv_x}{d\tau} = \text{Re} \sum_l h_{y,l} \exp(ikx + il\kappa z),
$$

$$
\frac{dp_z}{d\tau} = \text{Re} \sum_l \epsilon_{z,l} \exp(ikx + il\kappa z),
$$

$$
+ v_x \text{Re} \sum_l h_{y,l} \exp(ikx + il\kappa z).
$$

(14)

Let’s investigate a set of equations (14) on stability in a general dispersion equation shall keep only resonant members. \( \omega = \frac{n\Omega}{\sqrt{1 + n^2 \Omega^2}} \) also is in the correspondence consent with the theory. The polarization structure of radiation corresponds to radiation of a dipole. Power of the magnetron \( \sim 750 \text{ kW} \) (7.9 kV/cm). In accordance with \( \lambda = \frac{d}{\beta} \) it is necessary \( \sim 4.9 \text{ kV/cm} \). The maximum gain, reached in experiment, reached 20 dB.

5.2. NONLINEAR ANALYSIS

The numerical analysis of self-consistent set of equations (14) has confirmed the presence of instability in the considered system. System (14) was solved in two-wave approximation, that is at equations it was taking into account main mode with \( l = 0 \) and wave which is correspond first order of diffraction with \( l = 1 \). Conditions for the most effective excitation of wave for \( n = \mu = 10 \), \( n\Omega \approx \sqrt{1 + \omega_k^2 + \omega_n J_n(\mu)} \) also were selected. In these conditions the excitation of 10-th harmonics was observed. It is necessary to note, that it is carried out control calculations for \( l = -3, 0, 3 \). The result of calculations is in accord with two-wave approximation. Here amplitude of spatial harmonics reduces as \( \theta \rightarrow 0 \).

5.3. EXPERIMENTAL RESULTS

Experimental investigations \([11]\) were carried out in two-frequency range: SHF and UV. The excitation of oscillations for SHF range was observed only at simultaneous presence of oscillators produced by plasma electrons and periodical inhomogeneity produced by artificial grating, which immersed in plasma. If the grating or plasma were removed, the radiation on harmonics missed. Moreover, the plasma could be disposed from grating on different spacing interval. Thus there is some critical spacing interval \( \sim 2 \text{ mm} \), since which one of the signal on harmonics disappears. The directional diagram structure of radiation also is in the correspondence consent with the theory.

5.1. LINEAR APPROXIMATION.

DISPERSION EQUATION

Let’s investigate a set of equations (14) on stability in linear approximation. For these purposes we’ll present the mass, charge and density of oscillators.

$$
\text{Im} \delta = (q / 2) \sqrt{\omega_k \omega_n J_n}. \quad (16)
$$

Thus self-consistent system of equations (14) have unstable solution with increment (16).

Fig. 5. The microwave signal amplitude (the upper ray) and the radiation signal from the crystal (the lower ray)
These results are in the good agreement with our representations about the mechanism of radiation.

CONCLUSIONS

The main conclusion from theoretical considerations is that - in all cases at presence of periodic heterogeneity's of medium or periodic potential the oscillators will effectively excite high numbers of harmonics. Intensity of harmonic radiation at local maximum is high enough.

High efficiency of harmonics excitation by ensemble of nonrelativistic oscillators in SHF range is shown experimentally and it is shown the capability to convert energy of external 10 cm of radiation to energy of an ultraviolet radiation (10^-5 cm). Experimental data are in the good agreement with our representations about the mechanism of radiation.

Maximum of radiation will come in most cases on radiation with a wavelength: \( \lambda = d/\beta \).

These results allow to expect on creation of a new type of beam generators similar to free electron laser, key difference which one is the possibility of using of flows of nonrelativistic charged particles.

REFERENCES