SPIN-TRIPLET SUPERFLUIDITY OF NEUTRON MATTER WITH SKYRME FORCES IN STRONG MAGNETIC FIELD NEAR $T_c$

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A dense homogeneous superfluid pure neutron matter (SPNM) with the effective Skyrme forces (depending on the density $n$ of the neutrons) and with spin-triplet $p$-wave pairing (similar to $^3$He-$A_3$ and $^3$He-$A_2$) in a strong uniform static magnetic field $H$ is studied in the framework of a generalized non-relativistic Fermi-liquid theory. General formulas (valid for arbitrary parameterization of the Skyrme forces) are derived analytically for the phase transition temperatures $T_{c1,2}$ of the neutron matter from normal to superfluid states of $^3$He-$A_3$ and $^3$He-$A_2$ types, respectively. The functions $T_{c1,2}(H,n)$ are linear with respect to $H$ (up to sufficiently high magnetic fields) and are non-monotone functions of density. Figures for $(T_{c1}(H,n)-T_{c2}(H,n))/H$ are plotted in the range $0.8n_0 \lesssim n \lesssim 3n_0$ ($n_0=0.17$ fm$^{-3}$ is the saturation density of the symmetric nuclear matter) for selected RATP, SkO$^*$ and Gs parameterizations of the Skyrme forces which have different power dependences on density. Such phases of dense SPNM may exist in cores of magnetized neutron stars.

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1. INTRODUCTION

The superfluid phases of $^3$He and superfluid pure neutron matter (SPNM) (existing inside liquid core of neutron stars at subnuclear $n \lesssim n_0$ (where $n_0=0.17$ fm$^{-3}$ is the saturation density of the symmetric nuclear matter) and supernuclear ($n > n_0$) densities of neutrons; see, e.g., [1] and references therein) are important examples of superfluid Fermi liquids (SFLs) with spin-triplet pairing. Here we have investigated superdense SPNM with $p$-wave pairing of $^3$He-$A_2$ type in a steady homogeneous magnetic field $H$ and have used the generalized non-relativistic Fermi-liquid approach [2] for derivation nonlinear integral equations for the order parameter (OP) and effective magnetic field (EMF) $H_{eff}$ inside SPNM [4,5] which are valid at arbitrary temperatures from the interval $0 \leq T \leq T_c$ ($T_c$ is the normal–superfluid phase transition (PT) temperature). The effective Skyrme interaction between neutrons depending on the neutron density $n$ (see review [3] and Ref. [4, 5]) have been used. Further we have found analytically the approximate solutions of the obtained integral equations in the vicinity of the PT temperature $T_{c0}(n)$ of the NM to superfluid state with triplet $p$-wave pairing (without magnetic field) and have obtained the general approximate formulas for the PT temperatures $T_{c1,2}(H,n)$ for arbitrary parameterization of the Skyrmie interaction. These functions $T_{c1,2}(H,n)$ are linear with respect to the $H$ up to sufficiently high magnetic fields (but $H <\ll \varepsilon_F/|\mu_n|$, where $\varepsilon_F(n)$ is the Fermi energy of NM and $\mu_n < 0$ is the magnetic dipole moment of neutron). We have specified formulas for $T_{c1,2}(H,n)$ for the so-called RATP, SkO$^*$ and Gs parameterizations [6-8] of the Skyrme forces and the corresponding figures were plotted. These figures demonstrate the behavior of PT temperature splitting $(T_{c1}(H,n)-T_{c2}(H,n))/H$ as non-monotonic function of $n$ at the variation of neutron density in the permissible range, $0.8n_0 \lesssim n \lesssim 3n_0$ (where the non-relativistic Fermi-liquid theory is valid).

Other authors studied phase transitions of NM to superfluid states with triplet pairing without (see, e.g., Ref. [9-14]) and with the effect of magnetic field [15] within other approaches and using different nucleon-nucleon effective interactions inside NM with several simplifying assumptions (e.g., neglecting the dependence of neutron effective mass on NM density).

2. EQUATIONS FOR THE OP AND EMF FOR SPNM WITH THE SKYRME FORCES AND TRIPLET PAIRING

As is known [1], the OP for the so-called non-unitary phase (NU) of $^3$He-$A_3$ type with $p$-wave pairing has the form

$$\Delta_j^k(p) = (\Delta_d d_j + i\Delta_e e_j)\psi(p),$$

$$\psi(p) \equiv (\hat{m}^j + i\hat{n}^j)\hat{p}, \quad p \equiv \hat{p}.$$  \hspace{1cm} (1)

Here $\Delta_j(T) \equiv (\Delta(T) + \Delta^*(T))/2$, $d$ and $e$ are mutually orthogonal real unit vectors in spin space, $d \cdot e = 0$, $d^2 = e^2 = 1$; $\hat{m}$ and $\hat{n}$ are mutually orthogonal real unit vectors in orbital space, $\hat{m} \cdot \hat{n} = 0$, $\hat{m}^2 = \hat{n}^2 = 1$. The value $\eta(p) \equiv |\Delta(p) \times \Delta(p)|$ is not equal zero for NU phases of SFL or SPNM in particular. Note that the superfluid phase of $^3$He-$A_3$ type is realized at the condition with $\Delta_d = 0$, $\Delta_e \neq 0$.

We have chosen the effective Skyrme forces as the interaction between neutrons for SPNM with spin-triplet $p$-wave pairing in spatially uniform magnetic
field $H$. A set of coupled equations for the OP of the $^3$He $- A_2$ type and effective magnetic field (EMF) $H_{\text{eff}}$ inside SPNM is simplified in the case of Skyrme interaction because the normal Fermi-liquid Landau’s exchange amplitudes $F_l \approx 0$ only for $l = 0$ and $l = 1$ (in contrast to the superfluid $^3$He, when it is necessary to take into account in general case also the amplitudes $F_l \approx 0$ with $l > 1$, see e.g. [16]). As a result, using general formulas for anomalous and normal distribution functions of quasiparticles [16, 17] (neutrons) for SPNM in magnetic field we have derived a set of integral equations for $\xi(p)$ and $\Delta_\lambda$, $\Delta_\lambda^\ast$. In this case for SPNM $\xi(p) = \xi(p)H/H \equiv -\mu_n H_{\text{eff}}(p) \,(\mu_n \approx -0.60308 \cdot 10^{-17}$ MeV/G) is the magnetic dipole moment of neutron [18] and for $\xi(p)$ we have equation [4]:

$$\xi(p) = -\mu_n H + (r + sp^2) K_\lambda(\xi) + s K_\lambda(\xi).$$

Here $r = t_0'(t_0' - t_0)/6n^2$ and $s = (t_0' - t_0')/(4\hbar^2)$, $n \equiv y_0$ is density of neutron matter; $t_0' = t_0' \cdot (1 - x_0)$, $t_0' \equiv t_1' \cdot (1 - x_0)$, $t_0' \equiv t_2' \cdot (1 + x_0)$, $t_0' \equiv t_2' \cdot (1 - x_0)$ and $1/6 \leq \alpha \leq 1/3$ are parameters of the Skyrme interaction (cf. [3]). The functionals $K_\lambda(\xi)$ ($p=2$, 4) in Eq. (2) have the form:

$$K_\lambda(\xi) = \frac{1}{(2\pi)^3\hbar^4} \int_{p_{\text{min}}}^{p_{\text{max}}} dq x \int_0^1 dx \kappa(q,x),$$

where

$$\kappa(q,x) = \frac{z(q) + \xi(q)}{E_z(q,x^2)} \tanh \left( \frac{E_z(q,x^2)}{2T} \right) - \frac{z(q) - \xi(q)}{E_z(q,x^2)} \tanh \left( \frac{E_z(q,x^2)}{2T} \right),$$

$$E_z^2 = q^2 \Delta_\lambda(1-x^2) + (z(q) + \xi(q))^2, \quad z(q) = q^2/2m_n^\ast - \mu \quad (m_n^\ast \text{ is the effective mass of neutron,} \mu \text{ is the chemical potential). We have taken into account that for SPNM with pairing of the }^3\text{He} - A_2 \text{ type the OP can be written as } \Delta_\lambda(1-x^2) = q \Delta_\lambda(1-x^2), \text{ where functions } \Delta_\lambda(1-x^2) \text{ obey the following equations [4]:}

$$\Delta_\lambda(1-x^2) = -\Delta_\lambda(1-x^2) \frac{c_3}{8\pi^2\hbar^2} \int_{p_{\text{min}}}^{p_{\text{max}}} dq x \int_0^1 dx \left( \frac{E_z(q,x^2)}{2T} \right) \tanh \left( \frac{E_z(q,x^2)}{2T} \right)$$

($p_{\text{max}} \geq p_F$, $p_{\text{max}} - p_{\text{min}})/p_F \ll 1$, $p_F$ is the Fermi momentum). Here $c_3 \equiv t_3(1 + x_2)/\hbar^2 < 0$ is coupling constant leading to spin-triplet $p$-wave pairing of neutrons, which is expressed through the parameters $t_0$ and $x_2$ of the Skyrme interaction (cf. Ref. [3, 4]). Note that we consider here (in contrary to Ref. [4, 5]) a model of neutron Cooper pairing in a thin shell in the vicinity of the Fermi sphere.

This set of nonlinear integral Eqs. (2) and (6) for the EMF and OP give us the possibility to describe thermodynamics of superfluid non-unitary phases of $^3$He $- A_2$ type in dense SPNM with spin-triplet $p$-wave pairing in static uniform high magnetic field at arbitrary temperatures from the interval $0 \leq T \leq T_c(H)$. In general case these equations can’t be solved analytically and it is necessary to use numerical methods for their solving. But we can solve Eqs. (2), (6) using analytical methods in the limiting case, when the temperature $(T \lesssim T_{c0})$ is near the PT temperature $T_{c0}(n)$ of dense NM to superfluid state (it is the theme of the section 3).

### 3. SOLUTIONS OF EQUATIONS FOR THE OP AND EMF FOR DENSE SPNM NEAR $T_C$

The set of nonlinear integral Eqs. (2) and (6) [4] for the EMF and components of the OP was solved by analytical methods and as a result the approximate expressions were obtained for “reduced” (to dimensionless form) phase transition temperatures $T_{c12} \equiv T_{c12}/\epsilon_F$ ($\epsilon_F \equiv p_F^2/2m_n^\ast$ is the Fermi energy of neutrons) of NM to superfluid states of $^3$He $- A_2$ type with triplet $p$-wave pairing in high magnetic field (with spin projections of the Cooper pairs along and against the magnetic field direction):

$$t_{c12} \approx t_{c0} \left[ 1 \pm \frac{h}{I_0} (AI_4 + BI_6) \right],$$

($t_{c0} \equiv T_{c0}/\epsilon_F$). Functions $t_{c12}(a;h,y)$ depend on the cutoff parameter $a \equiv \epsilon_{\text{max}}/\epsilon_F - 1$, which is the upper limit in the integrals $I_0$, $I_4$, $I_6$ and in the integrals, which enter the structure of the functions $A$ and $B$ (their explicit form see below in Appendix A). Parameter $a$ was introduced to avoid divergences of integrals and from the physical point of view it corresponds to the energy restriction of the quasiparticles (neutrons) by the maximal energy, i.e., $\epsilon(p) = p^2/2m_n^\ast \leq \epsilon_{\text{max}}$, which is somewhat larger than the Fermi energy ($\epsilon_{\text{max}} \gtrsim \epsilon_F$), so that $a \ll 1$ for the effective Skyrme forces using here as the interaction between neutrons. For the validity of the Fermi-liquid theory the following inequalities should be true:

$$h \equiv \frac{\mu_n H}{\epsilon_F(y)} < a \ll 1, \quad t_{c0} \equiv T_{c0}(a;y)/\epsilon_F(y) < a \ll 1.$$  

These inequalities mean small “smoothing” of the Fermi distribution step-function due to the influence of external magnetic field $H$ and temperature $T \lesssim T_{c0}(a;y)$ on the neutron matter (where $y \equiv n/n_0$).
It was obtained the following equation for reduced temperature $t_{c0}$ of PT for NM to the superfluid state with triplet $p$-wave pairing without magnetic field:

$$0 = 1 + c\frac{m_n}{\lambda} I_0(a; y),$$

(8)

where the integral $I_0(a; y)$ has the form:

$$I_0(a; y) = \int_a^\infty dx \left(1 + x^2\right)^{-1/2} \tanh \left(\frac{x}{2t_{c0}}\right)$$

$$= \ell(a) + 2 \ln \left(\frac{a}{2t_{c0}}\right).$$

(9)

It was obtained the following approximate expression for the function $\ell(a)$ at $2t_{c0} < a < 1$ (neglecting by small terms of the order $a^5$, $a^6$ ...):

$$\ell(a) \approx b_0 + \frac{3a^2}{8} + \frac{3a^4}{256},$$

(10)

$$b_0 = 2 \left(1 - \frac{1}{9} + \frac{2}{75}\right) + 4 \sum_{k=2}^\infty (-1)^{k+1} E(2k) \approx 1.64932,$$

(11)

$$E(-x) = \int_x^\infty \frac{e^t}{t} dt.$$  

From (8-11) we get the general formula for $t_{c0}$:

$$t_{c0} = \frac{1}{2} \exp \left(\frac{\ell(a) + 2 \frac{1}{c_n} \frac{1}{y m_n}}{2}\right),$$

(12)

which is valid for all Skyrme parameterizations. Here $c_n m_n n_0 / 2 < 0$ is the dimensionless value depending on the Skyrme parameters $t_2$ and $x_2$ (see after Eq. (6)); $m_n \approx 939.56563$ MeV/c$^2$ is the mass of free neutron [18]. Formulas (8), (12) contain the effective neutron mass $m_n^*$, which depends on the density of NM $n = y n_0$ according to the formula:

$$\frac{m_n^*}{m_n} = 1 + \frac{2 n_0}{4 \hbar^2} \left[ 1 - x_1 + 3 z_4 (1 + x_2) \right].$$

(13)

Here $n \approx (m_n + m_n^*) / 2 \approx 938.91897$ MeV/c$^2$ is mean free nucleon mass [19]; parameters $t_1$, $t_2$, $x_1$, $x_2$ have specific values for each Skyrme parameterization.

Note that the Fermi energy of the pure NM with density $n = y n_0$ is defined by the formula:

$$\varepsilon_F(y) = \left(3 m_n^* y n_0\right)^{2/3} \frac{\hbar^2}{2 m_n^*} \approx 60.8601 y^{2/3} m_n^{1/3} \text{ MeV.}$$

(14)

The integrals $I_0$ and $I_2$ in the general formulas (7) for the functions $t_{c12}(a; h, y)$ are defined as:

$$I_0(a; t_{c0}) = \int_a^\infty dx \left(1 + x^2\right)^{3/2} \frac{d}{dx} \left(\tanh \left(\frac{\sqrt{2} t_{c0}}{x}\right)\right).$$

(15)

The phenomenon of superfluidity in a NM at high densities $n > 3 n_0$ (inside the fluid core of a neutron star) should be investigated in the framework of a relativistic...
approach and with different interpretation of the hadron matter structure (including mesons, quarks, and other possible constituents). Here we have used the non-relativistic generalized Fermi-liquid approach [2, 4] because the following inequalities are valid for the Fermi energy (14):

\[ \varepsilon_{F,\text{Skyrme}}(y) << m_{\text{Skyrme}} c^2 \lesssim m_e c^2 \approx 939.56563 \text{ MeV} \]

over the whole interval of NM density variation \( 0.8 y \lesssim 3.0 \) studied here for RATP, SkO’ and Gs variants of the Skyrme forces.

**APPENDIX A**

The functions \( A(\alpha; y, t_0) \), \( B(\alpha; y, t_0) \) (see Eq. (7)) have the form \( y = n/n_0 \):

\[
A(\alpha; y, t_0, (a, y)) = \frac{1}{D(\alpha; y, t_0)} \left[ 1 + d_{12} y \frac{m_\alpha^*}{m_\alpha} (y) \left( i_1(\alpha; t_0) - i_1(\alpha t_0) \right) \right],
\]

\[
B(\alpha; y, t_0, (a, y)) = \frac{i_1(\alpha; t_0)}{D(\alpha; y, t_0)} d_{12} y \frac{m_\alpha^*}{m_\alpha} (y) \times \left[ 1 + d_{12} y^a + d_3 \right],
\]

\[
D(\alpha; y, t_0, (a, y)) \approx 1 - y^{1/3} (d_1 y^a + d_3) \times \frac{m_\alpha^*}{m_\alpha} (y) i_1(\alpha; t_0) - 2 d_{12} y \frac{m_\alpha^*}{m_\alpha} (y) i_1(\alpha; t_0)
\]

\[
- d_{12} y^a \left( \frac{m_\alpha^*}{m_\alpha} (y) \right)^2 \left( i_1(\alpha; t_0) i_2(\alpha; t_0) - \xi(\alpha; t_0) \right).
\]

The coefficients \( d_{12}, d_3, d_0 \) take the following general form for all Skyrme parameterizations [3, 18]:

\[
d_{12} = \frac{m_e c^2}{(\hbar c)^2} \frac{3 n_0}{8} \cdot (t'_{12} - t'_{12}) \approx \left( t'_{12} - t'_{12} \right) \times 0.0015382753 \left( \frac{1}{MeV \cdot fm^3} \right),
\]

\[
d_3 = \frac{m_e c^2}{(\hbar c)^2} \left( \frac{3 n_0}{8 \pi} \right)^{1/3} \frac{m_\alpha^*}{m_\alpha} \frac{t'_{12}}{6} \approx t'_{12} \times 0.00290399843 \left( \frac{1}{MeV \cdot fm^3 + 3 \alpha} \right),
\]

\[
d_0 = \frac{m_e c^2}{(\hbar c)^2} \left( \frac{3 n_0}{8 \pi} \right)^{1/3} \cdot \xi' \approx t'_{12} \times 0.00290399843 \left( \frac{1}{MeV \cdot fm^3} \right)
\]

Here power index is \( 1/6 \leq \alpha \leq 1/3 \) and the Skyrme parameters are \( t'_{12} = t_0 (1 - x_0) \), \( t'_{12} = t_1 (1 - x_1) \), \( t_0' = t_0 (1 + x_0) \), \( t_1' = t_1 (1 - x_1) \) (see after Eq. (2)).

Integrals \( i_j(\alpha; t) \) \( (j = 1, 3, 5) \) in (A.1-3) are defined as:

\[
i_j(\alpha; t) \equiv \int_{-\infty}^{t} dx \frac{d}{dx} \tanh \left( \frac{x}{2t} \right).
\]

**APPENDIX B**

**FIGURES FOR SPNM WITH RATP-SKYRME FORCES**

\( \tau_{RATP} (a; y) \)

**Fig. 1.** Splitting \( \tau_{RATP} (a; y) \equiv (t_0 (\alpha; h, y) - t_3 (\alpha; h, y)) / h \) (in a magnetic field \( h \equiv |\mu_e| H < a \times 10^{-1} \)) of reduced phase transition (PT) temperatures of NM (with RATP parameterization of the Skyrme forces) to superfluid states of the \( ^3\text{He} - \Lambda_{12} \) type as a function of reduced density \( y = n/n_0 \) and small cutoff parameter a

\( \tau_{RATP} (0.1; y) \)

**Fig. 2.** PT temperature splitting \( \tau_{RATP} (0.1; y) \) for SPNM with RATP-Skyrme parameterization and p-wave pairing of the \( ^3\text{He} - \Lambda_{12} \) type in strong magnetic field as a function of reduced density y (cutoff \( a = 0.1 \))
Figures for SPNM with SKO'-Skyrme forces

Fig. 3. Splitting $\tau_{SKO}(a; y) \equiv \left( t_{11}(a; h, y) - t_{22}(a; h, y) \right)/h$
(in a magnetic field $h \equiv \frac{|\mu_{z}|}{\sigma_{F}(y)} < a < < 1$) of reduced PT temperatures of NM (with SKO'-Skyrme parameterization) to superfluid states of the $^3He - A_{1,2}$ type as a function of reduced density $y$ and small cutoff parameter $a$

Fig. 4. PT temperature splitting $\tau_{SKO}(0.1; y)$ for SPNM with SKO'-Skyrme parameterization and p-wave pairing of the $^3He - A_{1,2}$ type in strong magnetic field as a function of reduced density $y$ (cutoff $a = 0.1$)

Figures for SPNM with GS-Skyrme forces

Fig. 5. Splitting $\tau_{GS}(a; y) \equiv \left( t_{11}(a; h, y) - t_{22}(a; h, y) \right)/h$
(in a magnetic field $h \equiv \frac{|\mu_{z}|}{\sigma_{F}(y)} < a < < 1$) of reduced PT temperatures of NM (with GS-Skyrme parameterization) to superfluid states of the $^3He - A_{1,2}$ type as a function of reduced density $y$ and small cutoff parameter $a$

Fig. 6. PT temperature splitting $\tau_{GS}(0.1; y)$ for SPNM with GS-Skyrme parameterization and p-wave pairing of the $^3He - A_{1,2}$ type in strong magnetic field as a function of reduced density $y$ (cutoff $a = 0.1$)

References


ТРИПЛЕТНАЯ ПО СПИНУ СВЕРХТЕКУЧЕСТЬ НЕЙТРОННОЙ МАТЕРИИ С СИЛАМИ СКИРМА В СИЛЬНОМ МАГНИТНОМ ПОЛЕ ВБЛИЗИ $T_c$

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В рамках обобщенной нерелятивистской ферми-жидкостной теории изучается плотная однородная сверхтекучая чисто нейтронная материя (СНМ) с эффективными силами Скирма (звавящимися от плотности $n$ нейтронов) и с триплетным по спину $p$-спариванием (подбідним до $^3$He-A, и $^3$He-A2) в сильном постоянном однородном магнитном поле $H$. Аналитически введены общие формулы (сравнивые для произвольной параметризации сил Скирма) для температур фазового перехода $T_{c1,2}$ нейтронной материи из нормального в сверхтекучее состояние типа $^3$He-A, и $^3$He-A2, соответственно. Функции $T_{c1,2}(H,n)$ являются линейными по $H$ (включая достаточно сильные магнитные поля) и немонотонными функциями плотности. Построены графики функции $(T_{c1}(H,n)−T_{c2}(H,n))/H$ в интервале изменения плотности $0.8n_0≤n≤3n_0$ ($n_0=0.17$ фм$^{-3}$ — плотность насыщения симметричной ядерной материи) для выбранных RATP, SKO' и Gs параметризаций сил Скирма, которые имеют разную степенную зависимость от плотности. Такие фазы плотной СНМ могут существовать в сердцевинах намагниченных нейтронных звезд.

СПИНИ-ТРИПЛЕТНАЯ НАДПЛИННІСТЬ НЕЙТРОННОЇ МАТЕРІЇ З СИЛАМИ СКІРМА У СИЛЬНОМУ МАГНИТНОМУ ПОЛІ ПОБЛИЗУ $T_c$

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У рамках узагальненої нерелятивистської фермі-рідинної теорії вивчається густа однорідна надплинна суто нейтронна матерія (ННМ) з ефективними силами Скирма (що залежать від густини $n$ нейтронів) та зі спин-триплетним $p$-спариванням (подбідним до $^3$He-A1 та $^3$He-A2) у сильному постійному однорідному магнітному полі $H$. Аналітично введені загальні формулі (які справедливі для довільної параметризації сил Скірма) для температур фазового переходу $T_{c1,2}$ нейтронної матерії з нормальногого в надплинні стани типу $^3$He-A1 та $^3$He-A2, відповідно. Функції $T_{c1,2}(H,n)$ є лінійними по $H$ (включаючи достатньо сильні магнітні поля) та немонотонними функціями густини. Побудовані графики функції $(T_{c1}(H,n)−T_{c2}(H,n))/H$ в інтервалі зміни густини $0.8n_0≤n≤3n_0$ ($n_0=0.17$ фм$^{-3}$ — густина наслідка симметричної ядерної матерії) для відібраных RATP, SKO’ та Gs параметризацій сил Скірма, які мають різний показник степенної залежності від густини. Можливо, що такі фази густої ННМ існують у серцевинах намагниченних нейтронних зірок.