MULTIPLE COULOMB SCATTERING WITH FIXED ENTRANCE AND EXIT PARAMETERS IN FERMI APPROXIMATION

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The interest in the probability distribution function, which describes the position of a heavy charged particle and direction of motion in matter, while its initial and final coordinates and angles are known, relates to proton imaging that is currently developing actively. The modern approach to calculating proton trajectories and widths is based on an approximations (first made by Fermi) that were never explained or justified. In the present work, we study the origin of the above-mentioned approximations, present the full formula for the probability distribution function, and show a limitation of the method that appears in the case of full formula.

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1. INTRODUCTION

Modern interest in probability distribution function, that describes heavy charged particle position and direction of motion in matter, while its initial and final coordinates and angles are known, relates to proton imaging that is currently developing actively (see, for example, [1, 2]). The image quality depends strongly on the accuracy of the determination of the trajectory of a proton. Unfortunately, the process of proton interaction with matter differs significantly from X-rays, and there is no simple relation between the characteristics of the proton and its trajectory within an object. Due to multiple Coulomb scattering (MCS), these trajectories are highly complex and depend on a number of random parameters.

Currently, the approach of [3] is used to calculate the trajectory of the proton and its width (uncertainty in the trajectory). This approach is based on the Fermi formula and his method, which was proposed to estimate the spatial distribution of cosmic-ray particles [4], however this method contains approximations that were never explained or justified. Further investigations [5, 6, 7] were focused on a more accurate implementation of the MCS theory; however, the method of probability distribution function construction was not analysed.

In the present work, we study the origin of the above-mentioned approximations, present the full formula for the probability distribution function, and show the limitations of the method in the case of full formula. The correctness of the Fermi formula is not the subject of our study.

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2. APPROXIMATE PROBABILITY DISTRIBUTION FUNCTION

Owing to its random nature, MCS is described using statistical laws. The possible particle position can be represented as the probability of finding the particle at a specific point in space. This distribution function is given by the Fermi formula [4]

$$F(t,y,\theta)=\frac{\sqrt{3}}{2\pi}\frac{\omega^2}{t^2}exp\left[-\omega^2\left(\frac{\theta^2}{t}-\frac{3y\theta}{t^2}+\frac{3y^2}{t^3}\right)\right],$$

where y is the lateral displacement, θ is the angular deflection, t is the thickness, and ω is the parameter depending on the particle energy. All of the distances are measured in radiation lengths.

By integrating this function over y, one can obtain the function $G(t,\theta)$, which represents the angular distribution irrespective of the lateral displacement.

$$G(t,\theta)=\int_{-\infty}^{\infty}Fdy=\frac{1}{2\sqrt{\pi}}\frac{\omega}{\sqrt{t}}exp\left(-\frac{1}{4}\frac{\omega^2\theta^2}{t}\right).$$

Similarly, by integrating the function $F(t,y,\theta)$ over θ, one obtains the function $H(t,y)$, which gives the spatial distribution (lateral displacement), independent of the angle.

$$H(t,y)=\int_{-\infty}^{\infty}Fd\theta=\frac{\sqrt{3}}{2\sqrt{\pi}}\frac{\omega}{t^{3/2}}exp\left(-\frac{3}{4}\frac{\omega^2y^2}{t^3}\right).$$

A probability distribution function with a fixed exit displacement and angle is constructed by multiplying two Fermi functions, before and after the point where this distribution is sought [4].

The probability for a particle that starts at the point $t = 0, y = 0$ with $\theta = 0$ and goes through $y = y_0$ with $\theta = \theta_0$ at the point $t = t_0$ to reach the point $t = L, y = y_L$ with $\theta = \theta_L$ (Fig.1) is

$$P_A(t_0, y_0, \theta_0, L, y_L, \theta_L) = F(t_0, y_0, \theta_0)F(L-t_0, y_L-y_0-(L-t_0)\theta_0, \theta_L-\theta_0)\,. \quad (1)$$

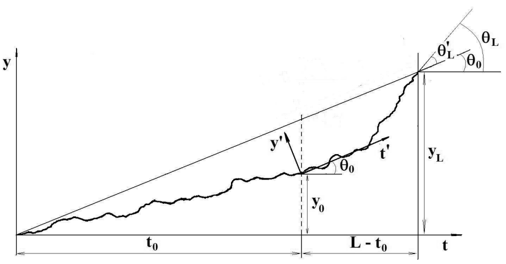

Fig.1. *Particle trajectory*

All particular cases: space or (and) angle distributions with a fixed exit displacement and (or) angle can be obtained by integrating this function with respect to the appropriate variables.

As is observed in Fig.1, the simplifications accepted in this formula relate to the second Fermi function. In the approximation of small θ_0 ($tg\theta_0 \sim \theta_0$), the distance along the axis *t'* is substituted with the distance along the axis *t*, and the displacement along the axis *y'* is substituted with the displacement along the axis *y*.

An absolutely correct probability distribution function with fixed exit displacement and angle $P(t_0, y_0, \theta_0, L, y_L, \theta_L)$ constructed on the basis of the Fermi function should satisfy the condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(t_0, y_0, \theta_0, L, y_L, \theta_L)dy_0 d\theta_0 = F(L, y_L, \theta_L)\,. \quad (2)$$

An approximate function should not naturally do so. However, function (1) satisfies equation (2) exactly. This is rather strange and requires a detailed analysis.

Further simplifications, which completely ignore θ_0 in the second Fermi function, lead to an absurd result

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_B(t_0, y_0, \theta_0, L, y_L, \theta_L)dy_0 d\theta_0 =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(t_0, y_0, \theta_0)F(L-t_0, y_L-y_0, \theta_L)dy_0 d\theta_0 =$$
$$\int_{-\infty}^{\infty} H(t_0, y_0)F(L-t_0, y_L-y_0, \theta_L)dy_0 =$$
$$\frac{\sqrt{3}}{2\pi}\frac{\omega^2}{\sqrt{(L-t_0)(4t_0^3+(L-t_0)^3)}} \times exp\left[-\omega^2\left(A\theta_L^2 - B\theta_L y_L + Cy_L^2\right)\right]\,;$$
$$A = \frac{L^3 - 3L^2 t_0 + 3Lt_0^2}{(L-t_0)(4t_0^3+(L-t_0)^3)}\,;$$
$$B = \frac{3(L-t_0)}{4t_0^3+(L-t_0)^3}\,; \qquad C = \frac{3}{4t_0^3+(L-t_0)^3}\,.$$

This result is not only unequal to the Fermi function at the point L, but it also depends on the indefinite parameter t_0, which would have to disappear after the integration.

3. FULL PROBABILITY DISTRIBUTION FUNCTION

The construction of the full function should include rotation of the coordinate system in the second Fermi function

$$P_F(t_0, y_0, \theta_0, L, y_L, \theta_L) = F(t_0, y_0, \theta_0)F(t', y', \theta') =$$
$$F(t_0, y_0, \theta_0)\times F(t\cos\theta_0 + y\sin\theta_0, -t\sin\theta_0 + y\cos\theta_0, \theta_L - \theta_0)\,,$$

if

$$t' = t\cos\theta_0 + y\sin\theta_0\,;$$

$$y' = -t\sin\theta_0 + y\cos\theta_0\,;$$

$$\theta' = \theta_L - \theta_0\,.$$

The probability of finding the particle at a distance $L > t_0$ in a non-rotated coordinate system is

$$P_F(t_0, y_0, \theta_0, L, y_L, \theta_L) =$$
$$F(t_0, y_0, \theta_0)\times$$
$$F((L-t_0)\cos\theta_0 + (y_L - y_0)\sin\theta_0, -(L-t_0)\sin\theta_0 + (y_L-y_0)\cos\theta_0, \theta_L - \theta_0)\,.$$

This formula shows that the approximations of [4] consist of transformations

$$(L-t_0)\cos\theta_0 + (y_L-y_0)\sin\theta_0 \to (L-t_0) \equiv \cos\theta_0 = 1, \sin\theta_0 = 0\,;$$
$$-(L-t_0)\sin\theta_0 + (y_L-y_0)\cos\theta_0 \to -(L-t_0)\theta_0 + y_L - y_0 \equiv \cos\theta_0 = 1, \sin\theta_0 = \theta_0\,.$$

Such transformations seem contradictory. However, the power series expansion of the function P_F, with respect to the variables θ_0 and y_0/t_0. shows that P_F is equal to P_A up to the terms of second order

$$P_F(t_0, y_0, \theta_0, L, y_L, \theta_L) = P_A(t_0, y_0, \theta_0, L, y_L, \theta_L) + O\left(\theta_0^2, \left(\frac{y_0}{t_0}\right)^2, \theta_0\frac{y_0}{t_0}\right)\,.$$

This means that the approximation of [4] is the result of correct mathematical expansion in small parameters. Clearly, the difference between P_F and P_A becomes more evident with the increase of θ_0 and y_0/t_0. In practice, the difference will increase with the decrease in ω (particle energy) and the passed distance (t_0) increase (Fig.2).

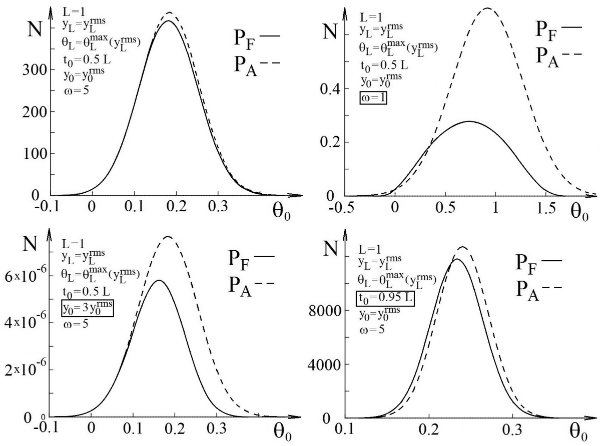

Fig.2. *P_F and P_A behaviour with the parameter change*

The particle trajectory is constructed by integrating P_F (or P_A) over θ_0. After integration, the difference in the distribution maximum (particle trajectory) and its width (uncertainty in the trajectory) not always can be ignored (Fig.3).

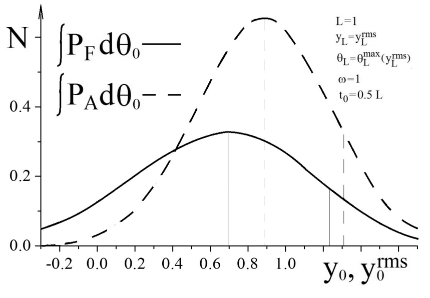

Fig.3. *P_F and P_A integrated over θ_0*

4. PROBLEMS WITH THE METHOD

The Fermi probability distribution function has its limitations. It only describes the probability in the "forward" direction, i.e. for $t > 0$. For $t < 0$, the distribution gives implausible results (Fig.4).

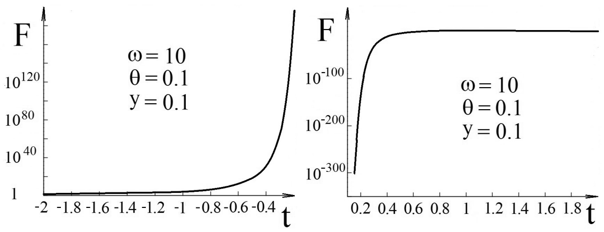

Fig.4. *Example of the Fermi function*

In the case of P_F, this means that there is an application limit for the second Fermi function (Fig.5)

$$t' > 0 \to (L - t_0)\cos\theta_0 + (y_L - y_0)\sin\theta_0 > 0\,. \quad (3)$$

The reason for this restriction is shown in Fig.6. For each y_0, there are values of θ_0 where t' becomes negative, and the second Fermi function gives an incorrect result.

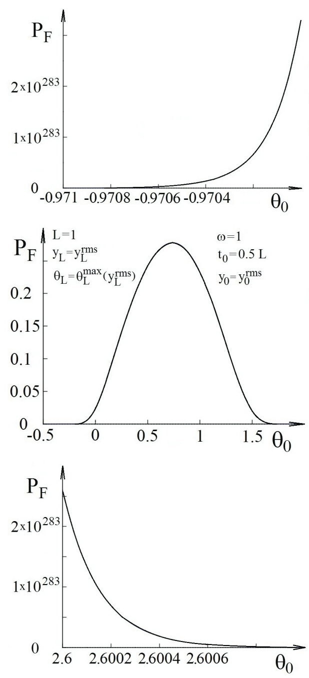

Fig.5. *Example of the full probability distribution function*

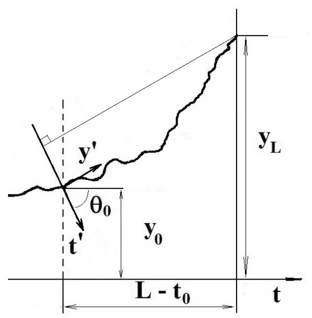

Fig.6. *Angle θ_0 with negative* t'

The θ_0 limits depend on the sign of $y_L - y_0$ (Fig.7)

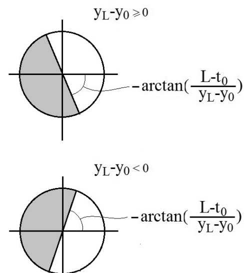

Fig.7. *Angle θ_0 with negative* t'

$$y_L - y_0 > 0, (-\alpha < \theta_0 < \pi - \alpha) \pm 2\pi n, n \in N\,;$$

$$y_L - y_0 < 0, (-\pi - \alpha < \theta_0 < -\alpha) \pm 2\pi n, n \in N\,,$$

where

$$\alpha = \arctan\left(\frac{L - t_0}{y_L - y_0}\right)$$

is the angle between the line $t = L$ and the straight line that connects the points (y_0,t_0) and (y_L,L).

This limit is especially important for the calculation of the particle trajectory. The integration of P_F over θ_0 can not be done from minus infinity to infinity, as it was in the case of P_A. The approximate probability distribution function has no such problem: $L - t_0 > 0$.

The same remark can be made about y_0. For each θ_0, there are y_0 where t' becomes negative, and corresponding limits can be calculated from (3).

Another problem of the method deals with sectors A and B in Fig.8. In these sectors, t' is positive, and the probability of finding the particle is small, but is non-zero. Clearly, further integration over y_0 and θ_0 will not nullify this probability. This indicates "backscattering". The particles first come to the point $t = t_0$ and then return to a point $t < t_0$. Such a process was not considered during the construction of the differential equation for Fermi function. It was assumed that $F(t+\Delta t, y, \theta)$ depends on $F(t, y, \theta)$ only ([4], page 266). Thus, the full probability distribution function is not equal to the Fermi probability distribution function, and it is not reasonable to expect realization of the equation (2).

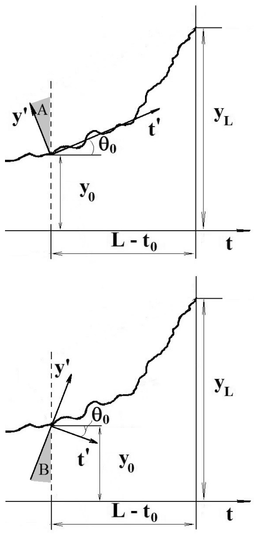

Fig.8. *Angle θ_0 with negative* t'

5. CONCLUSIONS

The analysis of the simplifications adopted earlier in the construction of the probability distribution function from two Fermi functions shows that it represents the correct mathematical expansion, with respect to small parameters, of the full probability distribution function.

Unlike the approximate, the full probability distribution function has a limitation related to the limitation of the Fermi formula itself.

When constructed with the proposed method, the full probability distribution function can not be considered identical to the Fermi probability distribution function.

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References

1. M. Scaringellaa, M. Brianzi, et al.// *Nucl. Instr. Meth.* 2013, v.A730, p.178-183.
2. S. Rit, G.Dedes, et al.// *Med. Phys.* 2013, v.40 (3) 031103.
3. U. Schneider and E. Pedroni // *Med. Phys.* 1994, v.21 (11), p.1657-1663.
4. B. Rossi and K. Greisen // *Rev. Mod. Phys.* 1941, v.13, p.240-311.
5. D.C. Williams// *Phys. Med. Biol.* 2004, v.49, p.2899-2911.
6. R.W. Schulte, S.N. Penfold, et al.// *Med. Phys.* 2008, v.35, p.4849-4856.
7. B. Erdelyi// *Phys. Med. Biol.* 2009, v.54, p.6095-6122.

МНОГОКРАТНОЕ КУЛОНОВСКОЕ РАССЕЯНИЕ С ФИКСИРОВАННЫМИ НАЧАЛЬНЫМИ И КОНЕЧНЫМИ ПАРАМЕТРАМИ В ПРИБЛИЖЕНИИ ФЕРМИ

Р. К. Л. Силва, В. Деняк, С. А. Пащук, У. Р. Счелин

Интерес к функции распределения вероятности, которая описывает положение и направление движения тяжёлой заряженной частицы в веществе в случае, когда её начальные и конечные координаты и углы известны, связан с методикой создания изображения при помощи протонов, которая активно развивается в настоящее время. Современный подход к расчёту траектории протона и её ширины основывается на приближении (впервые сделанном Ферми), которое никогда не было объяснено или обосновано. В настоящей работе мы исследуем происхождение вышеупомянутого приближения, приводим полную формулу для функции распределения вероятности и показываем ограничения метода, появляющиеся в случае полной формулы.

БАГАТОРАЗОВЕ КУЛОНІВСЬКЕ РОЗСІЯННЯ З ЗАФІКСОВАНИМИ ПОЧАТКОВИМИ І КІНЦЕВИМИ ПАРАМЕТРАМИ В НАБЛИЖЕННІ ФЕРМІ

Р. К. Л. Сілва, В. Деняк, С. А. Пащук, У. Р. Счелін

Зацікавленість у функції розподілу вірогідності, що описує положення і напрямок руху важкої зарядженої частинки у речовині у випадку, коли відомі її початкові і кінцеві координати і кути, пов'язана з методікою створення зображення за допомогою протонів, яка активно розвивається в наш час. Сучасний підхід до розрахунку траєкторії протона та її ширини базується на наближенні (вперше зробленому Фермі), яке ніколи не було роз'яснено або обгрунтовано. У даній роботі ми досліджуємо походження вищезгаданого наближення, наводимо повну формулу для функції розподілу вірогідності та показуємо обмеження метода, що з'являються у випадку повної формули.