THE DIFFUSION EFFECTS IN RELATIVISTIC ELECTRON BEAM IN AN UNDULATOR

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We consider diffusion processes in momentum space of a relativistic electron beam moving in a spatially periodic magnetic field of an undulator. Basing on the dynamics of individual particles motion under the action of the pair interaction forces the longitudinal diffusion coefficient has been derived. The conditions for the high-gain self-amplification of spontaneous radiation in ultrashort-wavelength FELs have been discussed.

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INTRODUCTION

As it is known, relativistic electron beams, moving in a spatially periodic static magnet field (undulator) are the sources of intense narrowband electromagnetic radiation. The wavelength of this radiation is proportional to the period of an external magnetic field and inversely proportional to the square of energy of electron. Such mechanism of interaction between ultra-relativistic electrons and external periodic magnetic field has been used to obtain the electromagnetic radiation in nanometer range of wavelengths by now [1 - 3].

At a spontaneous incoherent radiation of electromagnetic waves by relativistic electrons, moving in an undulator, there is a change of the average momentum of electrons, as a result of braking by the force of radiation friction. Moreover, influence of incoherent electromagnetic field of spontaneous radiation of individual electrons leads to the increase in root-mean-square spread in momenta in a relativistic electronic beam, moving in an undulator [4, 5]. The study of motion dynamics of electrons at the stage of spontaneous incoherent radiation is of interest regarding the researches directed on creation of sources of coherent electromagnetic radiation in X-ray range of wavelengths by means of relativistic electron beam passing through an undulator.

The interaction of initially monoenergetic electron beam with an undulator field has been considered in [5] and the expression describing the change of a root-mean-square longitudinal momentum of electrons, in the case when the spread in energy of electrons at the entrance of the undulator can be neglected, has been found. The motion of the beam of electrons, having at the entrance of the undulator some initial spread in longitudinal momentum, is considered in the given work. In the limit case of small value of the undulator parameter the expression for the diffusion coefficient in momentum space is obtained, which can describe both the initial stage of prebrownian motion of electrons in the electromagnetic field of undulator radiation, when approach of a monoenergetic electron beam is valid, and in the case of kinetic stage of particles diffusion.

1. PROBLEM STATEMENT

Let's consider a beam of relativistic electrons, moving in the spatially periodic static magnet field of helical undulator

$$\mathbf{H}_u = H_0\left[\mathbf{e}_x \cos(k_u z) + \mathbf{e}_y \sin(k_u z)\right], \qquad (1)$$

$k_u = 2\pi/\lambda_u$, H_0 and λ_u are the amplitude and period of magnetic field, $\mathbf{e}_x, \mathbf{e}_y$ are the unit vectors along axes x and y the Cartesian system of coordinates.

Moving in an undulator, electrons radiate. The electric field produced by individual electron (s-th) in undulator can be found from formulas for the field of a charge, moving with acceleration [6].

$$\mathbf{E}_s = e\frac{\left(\mathbf{n}'_s - \boldsymbol{\beta}'_{s s}\right)}{\gamma_s'^2 R_s'^2 \left(1-\mathbf{n}'_s\boldsymbol{\beta}'\right)^3} + \frac{e\left[\mathbf{n}'_s\left[\left(\mathbf{n}'_s - \boldsymbol{\beta}'_{s s}\right)\dot{\mathbf{v}}'_s\right]\right]}{c^2 R'_s\left(1-\mathbf{n}'_s\boldsymbol{\beta}'_s\right)^3}, \quad (2)$$

$$\mathbf{H}_s(\mathbf{r}, t; x_s) = \left[\mathbf{n}'_s \mathbf{E}_s\right], \qquad (3)$$

where $\mathbf{n}'_s = \mathbf{R}'_s / R'_s$, $\mathbf{R}_s = \mathbf{R}_s(t) = \mathbf{r} - \mathbf{r}_s(t)$, $\boldsymbol{\beta} = \mathbf{v}/c$, $\dot{\mathbf{v}} = d\mathbf{v}/dt$, $\gamma = \left(1-\beta^2\right)^{-1/2}$, c is the speed of light in vacuum, e is the electron charge, the prime denotes the values taken in retarding time t′, defined by the equation: $t' = t - R_s(t')/c$.

Considering the motion of a test particle in an undulator the equation describing its motion is possible to be written down in the form

$$\frac{d\,\mathrm{p}_{zi}}{dt} = \sum_s F_z^{(s)}\left[x_i(t), t; x_s\right], \ \frac{d\mathbf{r}_i}{dt} = \frac{\mathbf{p}_i(t)}{m\gamma_i(t)}, \qquad (4)$$

$$F_z^{(s)}(x, t; x_s) = e\left\{\mathrm{E}_{zs}(x, t; x_s) + \frac{1}{c}\left[\mathbf{v}\,\mathbf{H}_s(x, t; x_s)\right]_z\right\}, \quad (5)$$

where $F_z^{(s)}(x_i,t;x_s)$ is the longitudinal component of pair interaction force of two electrons, m is the mass of electron, $x_s(t) \equiv \{\mathbf{r}_s(t), \mathbf{p}_s(t)\}$ set of the Cartesian coordinates and momentum of s-th electron.

2. DIFFUSION COEFFICIENT

Distribution of the electrons in the beam at the entrance of the undulator is random, therefore the total electromagnetic field of radiation by individual electrons at the initial stage is incoherent. Assuming that at initial instant of time the motion of the electrons is uncorrelated, and there are many electrons in the beam, the diffusion coefficient in a longitudinal momentum can be taken from the equations of test electron motion [5, 7]

$$D_z = \frac{d}{2dt}\left\langle\left(\Delta p_{zi}\right)^2\right\rangle = \int_0^\tau d\tau' \int_{\Omega q} F_z^{(1)}\left[x_i^{(0)}(t), t; x_s^{(0)}(t, q_{0s})\right]\times$$
$$\times F_z^{(1)}\left[x_i^{(0)}(t-\tau'), \tau'; x_s^{(0)}(t-\tau', q_{0s})\right] f_1(q_{0s})\, \mathrm{v}_z(t_{0s})\, dq_{0s}\,, \qquad (6)$$

where $q_{0s}=(\mathbf{p}_{0s}, x_{0s}, y_{0s}, t_{0s})$, $dq_{0s}=d\mathbf{p}_{0s}dx_{0s}dy_{0s}dt_{0s}$, $\tau=t-t_{0i}$, f_1 is the single-particle distribution function, $x_i^{(0)} = \left(\mathbf{r}_i^{(0)}, \mathbf{p}_i^{(0)}\right)$. As we consider time intervals τ small in comparison with the time of the significant change of the electrons trajectory, in the right-hand side of Eq. (6), in the pair interaction force, we have replaced coordinates and the momenta with unperturbed trajectory $\mathbf{r}^{(0)}(t)$ and momenta $\mathbf{p}^{(0)}(t)$ of electron in the undulator.

Let's assume that the electron beam is cylindrical with radius r_b and constant average density of electrons n_b for $r \le r_b$, and at the initial time (at z=0) distribution function in momenta takes the form:

$$f(\mathbf{p}) = \frac{n_b}{\sqrt{2\pi} p_{th}} \delta(\mathbf{p}_\perp) \exp\left[-\frac{(p_z - p_{z0})^2}{2p_{th}^2}\right].$$

The equilibrium velocity and trajectory of electron in the field (1) are:

$$\mathbf{r}_s^{(0)}(t) = \mathbf{r}_{0s} + \mathbf{v}_{0s}(t - t_{os}) - \mathbf{e}_x r_{us} \sin k_u z_s(t) + \mathbf{e}_y r_{us} \cos k_u z_s(t),$$

$$\mathbf{v}_s^{(0)}(t) = \mathbf{v}_{0s} - \mathbf{e}_x \mathrm{v}_{\perp s} \cos k_u z_s(t) - \mathbf{e}_y \mathrm{v}_{\perp s} \sin k_u z_s(t),$$

where $r_u = \dfrac{cK}{k_u \mathrm{v}_{z0} \gamma_0}$, $\mathrm{v}_\perp = \dfrac{cK}{\gamma_0}$, $K = \dfrac{|e| H_0}{mc^2 k_u}$, $z_s(t) = \mathrm{v}_{z0}(t - t_{0s})$.

In the case of small undulator parameter K<<1, considering only the second term of force (5), from Eqs. (2), (3) and (4) we get the expression for pair interaction force between electrons:

$$F_{zs}\left[\mathbf{r}_i^{(0)}(t), t; q_{0s}\right] = -\left(e \mathrm{K} k_u \beta_s \gamma_s^2/\gamma_i\right)^2 G(\Delta z_{si}, \rho_{si}/\gamma_s), \quad (7)$$

$$G(x,y) = \frac{1}{k_{0s} R_*}\left[\left(\beta_s + \frac{x}{R_*} - \frac{\beta_s}{k_{0s}^2 R_*^2}\right)\sin\psi + \left(\beta_s + \frac{x}{R_*}\right)\frac{\cos\psi}{k_{0s} R_*}\right],$$

where $\psi(x,y) = k_u \gamma_s^2 (x + \beta_s R_*)$, $R_*(x,y) = \left(x^2 + y^2\right)^{1/2}$, $\Delta z_{si} = z_i - \mathrm{v}_{zs}(t - t_{0s})$, $\rho_{si} = |\mathbf{r}_{\perp i} - \mathbf{r}_{0\perp s}|$, $k_{0s} = \beta_{0s} \gamma_{0s}^2 k_u$.

Let's substitute expression for force (7) in the equation (6). Assuming that change in momentum occurs at a distance greater than a period of the undulator, in expression (7) we retain the terms inversely proportional to the first degrees R_*. We will also consider that the basic contribution to the integral (6) will give terms containing a difference of phases at time t and t-τ′. Then Eq. (6) can be reduced to the form:

$$D_z = \int_0^\tau d\tau' K_z(t, \tau'), \quad (8)$$

$$K(t,\tau') = \pi\left(e^2 K^2 k_u \gamma_0\right)^2 \int d\mathbf{p}_{0s} f(\mathbf{p}_{0s}) \int dV_{0s} \frac{\cos\psi_-}{r' |\mathbf{r}' + \mathbf{w}|}, \quad (9)$$

where $\mathbf{r}' = ((\mathbf{r}_{\perp i} - \mathbf{r}_{\perp s})/\gamma_{0s}) + (z_i - z_s)$, $\mathbf{w} = (\mathbf{v}_i - \mathbf{v}_s)\tau'$, $\psi_- = \psi(t) - \psi(t - \tau')$, $dV_{0s} = \mathrm{v}_{0s}\, dx_{0s} dy_{0s} dt_{0s}$.

The limits of integration in the Eq. (9) are defined by the time of radiation propagation from electron-radiator (s-th) to considered test electron and the transverse dimensions of the beam:

$$\Delta z_{is} + \beta_{0i} R_* \le z/\gamma_i^2, \quad r_{\perp s} = \left(x_{os}^2 + y_{0s}^2\right)^{1/2} \le r_b. \quad (10)$$

In the right-hand side of Eq. (9) at the integration over initial coordinates it is expedient to transform to the new variables r', θ, φ:

$$x_{os} - x_{oi} = \gamma_s r' \sin\theta \cos\varphi, \quad y_{os} - y_{oi} = \gamma_s r' \sin\theta \sin\varphi, \quad \Delta z_{si} = r' \cos\theta.$$

Let's find the diffusion coefficient for electrons moving near to the beam axis. Thus the range of the integration on r' and θ according to the Eq. (10) is:

$$r'(\cos\theta + \beta_i) \le z'/\gamma_i^2, \quad \gamma_s r' \sin\theta \le r_b. \quad (11)$$

In Eq. (9) we will consider the forces exerted on the test electron by the electrons, moving behind it $z_s<z_i$ ($0<\theta<\pi/2$). Integrating in Eq.(9) on r' and θ at $z>z_*$, and substituting the obtained expression in Eq.(8), we find:

$$D_z = \pi\left(e^2 K^2 k_u\right)^2 r_b \gamma_0 n_b / \mathrm{v}_{zi} \int_0^{z-z_*} e^{-a^2 x^2} \cos bx\, dx, \quad (12)$$

where $a = \dfrac{p_{th} k_u}{\sqrt{2} p_{z0}}$, $b = k_u \dfrac{p_{zi} - p_{z0}}{p_{z0}}$, $z_* = \gamma_0 r_b$.

Using this formula, it is possible to find the diffusion coefficient in momentum space for the various times at the certain initial energy spread of electrons.

3. DISCUSSION

From the expression (12), connecting the correlation function and diffusion coefficient, it follows that the correlation function can be written in the form:

$$K(t,t') = \pi\left(e^2 K^2 k_u\right)^2 r_b \gamma_0 n_b e^{-(\tau/\tau_c)^2} \cos(b \mathrm{v}_{0z} \tau), \quad (13)$$

where $\tau_c = \sqrt{2} p_{z0} / (p_{th} k_u v_{z0})$.

From (13) we see that correlation function oscillates on τ with decreasing amplitude for large values of t. For τ→∞ in formula (13) the correlation function tends to zero. Such dependence of correlation function on time describes chaotization of particles motion. Characteristic time of particles motion chaotization is τ_c, which is equal to the displacement time in the longitudinal direction, as a result of thermal motion, at the distance equal to the half of the wavelength of undulator radiations $\tau_c = 0.5\lambda/\mathrm{v}_{th}$, $\lambda = \lambda_u / 2\gamma_0^2$. For $\tau \gg \tau_c$ the motion of particles becomes chaotic.

The expression (12) describes the change of root-mean-square value of the momentum of electrons also at times $\tau \ll \tau_c$. The expression for diffusion coefficient in this case becomes:

$$D_z(p_{zi}) = \pi\left(e^2 K^2 k_u\right)^2 \frac{r_b \gamma_0 n_b}{\mathrm{v}_{zi}} z.$$

In this limiting case the change in time of root-mean-square value of the longitudinal momentum is described by the formula

$$\left[\left\langle (\Delta p_z)^2 \right\rangle\right]^{1/2} = (F_z)_R \tau \sqrt{N \frac{9\gamma_{z0} r_b}{2\lambda_u}}, \quad (14)$$

which coincides with the corresponding formula of [5], where $(F_z)_R = (2/3) r_0^2 H_0^2 \gamma_0^2 \beta_{z0}^3$, $N = n_b \lambda_u^3 / 8\gamma_{z0}^4$. For $\tau \ll \tau_c$ the motion of particles occurs under the influence of pair interaction forces of particles, the change in time of which is negligible. Therefore, the r.m.s. value of momentum is proportional to the time.

For $\tau \gg \tau_c$ the motion of electrons is random. The expression for diffusion coefficient becomes:

$$D_z = e^4 K^4 k_u \frac{r_b \gamma_0 n_b}{\mathrm{v}_{zi}} \frac{\pi^{3/2} p_{z0}}{\sqrt{2} p_{th}} \exp\left[-\frac{(p_{zi} - p_{z0})^2}{2p_{th}^2}\right]. \quad (15)$$

The r.m.s. value of the momentum increases proportionally to the square root of time. Such dependence of momenta spread in time describes the completely chaotic motion of particles.

The distance in the undulator at which the particles motion chaotization occurs can be written as $z_c = \mathrm{v}_{z0}\,\tau_c$. Then, for the electron beam with some initial energy spread, we find:

$$z_c = \sqrt{2}\,p_{z0}/(p_{th}k_u).$$

Thus, for the momentum spread at the entrance of the undulator p_{th} so that $p_{th}k_u z/p_{z0} < 1$ the r.m.s. deviation of the longitudinal momentum from equilibrium value increases proportionally to the distance traversed by electrons in the undulator (14). In this case the monoenergetic beam approximation [5] is applicable. In the opposite limit of large energies spread $p_{th}k_u z/p_{z0} \gg 1$ the kinetic stage of the radiative relaxation of an electron beam in the undulator occurs. At this stage the r.m.s. spread increases proportionally to the square root of time (15).

As we see from (14), at the initial energy spread for which the mode of self amplification of spontaneous undulator radiations occurs [8, 9], the energy spread in the beam can increase as a result of the radiative relaxation [5]. At small momentum spread $p_{th}k_u z/p_{z0} < 1$ the analysis of radiation formation in the mode of self amplification of spontaneous emission needs to be carried out while taking into account the effect of the radiative relaxation of the beam.

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ЭФФЕКТЫ ДИФФУЗИИ В РЕЛЯТИВИСТСКОМ ЭЛЕКТРОННОМ ПУЧКЕ В ОНДУЛЯТОРЕ

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Рассмотрены процессы диффузии в пространстве импульсов релятивистского электронного пучка, движущегося в пространственно периодическом магнитном поле ондулятора. Основываясь на динамике движения отдельных частиц под действием сил парного взаимодействия, получен продольный коэффициент диффузии. Обсуждаются условия реализации интенсивного самопроизвольного усиления спонтанного излучения в ультракоротковолновых ЛСЭ.

ЕФЕКТИ ДИФУЗІЇ В РЕЛЯТИВІСТСЬКОМУ ЕЛЕКТРОННОМУ ПУЧКУ В ОНДУЛЯТОРІ

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Розглянуто процеси дифузії в просторі імпульсів релятивістського електронного пучка, що рухається в просторово періодичному магнітному полі ондулятора. Ґрунтуючись на динаміці руху окремих частинок під дією сил парної взаємодії, отримано поздовжній коефіцієнт дифузії. Обговорюються умови реалізації інтенсивного самочинного посилення спонтанного випромінювання в ультракороткохвильових ЛВЕ.