

ONE-DIMENSIONAL INVERSE POWER REFLECTIONLESS

$$\text{POTENTIALS } V(x) = \text{const} \cdot |x - x_0|^{-n}$$

S.P. Maydanyuk

National Academy of Sciences of Ukraine, Institute for Nuclear Research, Kiev, Ukraine

e-mail: maidan@kinr.kiev.ua

A condition, at which inverse power one-dimensional potential $V(x) = \alpha / |x - x_0|^n$ ($\alpha = \text{const}$, $x_0 = \text{const}$, $x \in]-\infty; +\infty[$, n is a natural number) becomes reflectionless during propagation through it of a plane wave, is obtained on the basis of SUSY QM methods. A scattering of a particle on spherically symmetric potential $V(r) = \pm \alpha / |r - r_0|^n$ is analysed with taking into account of the reflectionless possibility.

PACS: 03.65.-w, 03.65.Db, 03.65.Nk, 03.65.Xp, 24.30.-v

1. INTRODUCTION

Methods of supersymmetric quantum mechanics (SUSY QM) allow finding quantum systems (both in the region of continuous energy spectrum and discrete one), which potentials have a penetrability coefficient of particles through them equal to one. One can name such quantum systems (and their potentials) as *reflectionless* [1].

A resonant tunneling phenomenon and, especially, papers, directed to study of its demonstration in concrete physical problems (for example, see Ref. [2]), have been caused an increased interest. The penetrability coefficient of the barrier during the resonant tunneling becomes large to the maximum. But the reflectionless potentials are interested in that they have the penetrability coefficient, practically equal to one in a whole region of the energy spectrum, whereas the resonant tunneling exists at selected energy levels only. A number of papers devoted to study of properties of the reflectionless quantum systems have been increasing each year. Here, note the bright reviews [3, 4], where both the methods for detailed study of properties of one- and multichannel reflectionless quantum systems, and enough simple approaches for their qualitative understanding are presented. All these methods have found their application in scattering theory (both in direct problem and in inverse one).

Note, that SUSY QM methods for study of the properties of the systems in the continuous energy spectrum are developed less than in the discrete one. Besides, majority of the obtained reflectionless potentials are expressed with use of series in enough complicated form, and any found reflectionless potential with a simple analytical form can be useful by its clearness in qualitative analysis of the quantum systems properties. In this paper we analyse the one-dimensional and spherically symmetric quantum systems in the region of the continuous energy spectrum, which potentials have an inverse power dependence on a space

coordinate, and we obtain conditions, when these systems (and potentials) become reflectionless.

2. INTERDEPENDENCE BETWEEN SPECTRAL CHARACTERISTICS OF POTENTIALS-PARTNERS

In the beginning we consider an one-dimensional case of a motion of a particle with mass m inside a potential field $V(x)$. Let's introduce the following operators A and A^+ :

$$\begin{aligned} A &= \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \\ A^+ &= -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \end{aligned} \quad (1)$$

where $W(x)$ is a function, given on the whole space region x . We suppose that this function is continuous on the whole region of its definition except for some possible points of discontinuity. On the basis of operators A and A^+ one can construct two Hamiltonians for a motion of this particle inside two different fields $V_1(x)$ and $V_2(x)$:

$$\begin{aligned} H_1 &= A^+ A = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x), \\ H_2 &= A A^+ = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_2(x), \end{aligned} \quad (2)$$

where potentials $V_1(x)$ and $V_2(x)$ are defined as follows:

$$\begin{aligned} V_1(x) &= W^2(x) - \frac{\hbar}{\sqrt{2m}} \frac{dW(x)}{dx}, \\ V_2(x) &= W^2(x) + \frac{\hbar}{\sqrt{2m}} \frac{dW(x)}{dx}. \end{aligned} \quad (3)$$

In development of SUSY QM theory the function $W(x)$ is named as *superpotential*, whereas the potentials $V_1(x)$ and $V_2(x)$ are named as *supersymmetric potentials-partners* [5]. Composition of Hamiltonians of two quantum systems on the basis of the operators A and A^+ establish interdependence between spectral characteristics (spectra of energy, wave functions) of

these systems. One can see a reason of such interdependence in that two different potentials $V_1(x)$ and $V_2(x)$ express through the same function $W(x)$.

If the energy spectra of these systems are discrete, then one can write:

$$\begin{aligned} H_1 \varphi_n^{(1)} &= A^+ A \varphi_n^{(1)} = E_n^{(1)} \varphi_n^{(1)}, \\ H_2 \varphi_n^{(2)} &= A A^+ \varphi_n^{(2)} = E_n^{(2)} \varphi_n^{(2)}, \end{aligned} \quad (4)$$

where $E_n^{(1)}$ and $E_n^{(2)}$ are the energy levels with number n (n is a natural number) for two systems with potentials $V_1(x)$ and $V_2(x)$, $\varphi_n^{(1)}$ and $\varphi_n^{(2)}$ are wave functions corresponding to these levels. We obtain:

$$\begin{aligned} H_2(A \varphi_n^{(1)}) &= A A^+ A \varphi_n^{(1)} = E_n^{(1)}(A \varphi_n^{(1)}), \\ H_1(A^+ \varphi_n^{(2)}) &= A^+ A A^+ \varphi_n^{(2)} = E_n^{(2)}(A^+ \varphi_n^{(2)}). \end{aligned} \quad (5)$$

We displace $V_1(x)$ by such a way that $E_0^{(1)} = 0$ (it has no influence into levels distribution inside energy spectra and into a form of wave functions). Analysing Eq. (5), one can obtain the following interdependencies between the energy spectra and the wave functions [5]:

$$\begin{aligned} E_n^{(2)} &= E_{n+1}^{(1)}, \\ E_0^{(1)} &= 0, \\ \varphi_n^{(2)} &= (E_{n+1}^{(1)})^{-1/2} A \varphi_{n+1}^{(1)}, \\ \varphi_{n+1}^{(1)} &= (E_n^{(2)})^{-1/2} A^+ \varphi_n^{(2)}. \end{aligned} \quad (6)$$

Here, a normalisation condition for wave functions inside the discrete energy spectrum are taken into account:

$$\int |\varphi_n^{(1)}(x)|^2 dx = 1, \quad \int |\varphi_n^{(2)}(x)|^2 dx = 1. \quad (7)$$

If the energy spectra of two systems are continuous, then one can find interdependence between their wave functions also (here, the expressions (5) will be changed a little):

$$\begin{aligned} \varphi^{(2)}(k, x) &= \text{const} \cdot A \varphi^{(1)}(k, x), \\ \varphi^{(1)}(k, x) &= \text{const} \cdot A^+ \varphi^{(2)}(k, x), \end{aligned} \quad (8)$$

where $\varphi^{(1)}(k, x)$ and $\varphi^{(2)}(k, x)$ are the wave functions for two systems with potentials $V_1(x)$ and $V_2(x)$. For obtaining the exact dependence between the wave functions in Eq. (8) one need to take into account a condition of their normalisation (for the continuous energy spectrum) with view of boundary conditions.

For the quantum systems with the continuous energy spectra the SUSY QM methods allow to establish the interdependence between the coefficients of the penetrability and the reflection [5]. Let the potentials $V_1(x)$ and $V_2(x)$ be finite at $x \rightarrow \pm\infty$, i.e. at

$$\text{for } x < 0 \quad \begin{cases} V_1(x) = W^2(x) - \frac{\square}{\sqrt{2m}} \frac{dW(x)}{dx} = \frac{\alpha}{(x-x_0)^2} \left(\alpha - \frac{\square}{\sqrt{2m}} \right), \\ V_2(x) = W^2(x) + \frac{\square}{\sqrt{2m}} \frac{dW(x)}{dx} = \frac{\alpha}{(x-x_0)^2} \left(\alpha + \frac{\square}{\sqrt{2m}} \right), \end{cases} \quad (16)$$

$$\text{for } x > 0 \quad \begin{cases} V_1(x) = W^2(x) - \frac{\square}{\sqrt{2m}} \frac{dW(x)}{dx} = \frac{\alpha}{(x+x_0)^2} \left(\alpha - \frac{\square}{\sqrt{2m}} \right), \\ V_2(x) = W^2(x) + \frac{\square}{\sqrt{2m}} \frac{dW(x)}{dx} = \frac{\alpha}{(x+x_0)^2} \left(\alpha + \frac{\square}{\sqrt{2m}} \right), \end{cases} \quad (17)$$

$$W(x \rightarrow \pm\infty) = W_{\pm} \quad (9)$$

we obtain:

$$V_1(x \rightarrow \pm\infty) = V_2(x \rightarrow \pm\infty) = W_{\pm}^2. \quad (10)$$

Consider propagation of a plane wave e^{ikx} in positive direction of x -axis in the field of the potentials $V_1(x)$ and $V_2(x)$. In result of its incidence from the left we obtain transmitted waves $T_1(k') e^{ik'x}$ and $T_2(k') e^{ik'x}$, and also reflected waves $R_1(k) e^{-ikx}$ and $R_2(k) e^{-ikx}$. We have:

$$\begin{aligned} \varphi^{(1,2)}(k, x \rightarrow -\infty) &\rightarrow e^{ikx} + R_{1,2} e^{-ikx}, \\ \varphi^{(1,2)}(k, x \rightarrow +\infty) &\rightarrow T_{1,2} e^{ik'x}, \end{aligned} \quad (11)$$

where k and k' are defined as follows:

$$k = \sqrt{E - W_-^2}, \quad k' = \sqrt{E - W_+^2}. \quad (12)$$

Taking into account the interdependence (8) between the wave functions for two systems with the continuous spectra, we write:

$$\begin{aligned} e^{ikx} + R_1 e^{-ikx} &= N((-ik + W_-) e^{ikx} + \\ &+ (ik + W_-) e^{-ikx} R_2), \end{aligned} \quad (13)$$

$$T_1 e^{ik'x} = N(-ik' + W_+) e^{ik'x} T_2,$$

where N is constant, defined from the normalisation conditions. Equating terms with the same exponent and estimating N , we obtain:

$$\begin{aligned} R_1(k) &= R_2(k) \frac{W_- + ik}{W_- - ik}, \\ T_1(k) &= T_2(k) \frac{W_+ - ik'}{W_- - ik}. \end{aligned} \quad (14)$$

Expressions (14) establish the interdependence between the amplitudes of penetrability and reflection for two quantum systems. The coefficients of penetrability and reflections of the potentials $V_1(x)$ and $V_2(x)$ can be calculated as squares of modules of the penetrability and reflection amplitudes.

3. POTENTIAL OF THE FORM

$$V(x) = \text{const}/(x - x_0)^2$$

Let's consider superpotential of the form:

$$W(x) = \begin{cases} \frac{\alpha}{x - x_0}, & \text{at } x < 0, \\ \frac{\alpha}{x + x_0}, & \text{at } x > 0, \end{cases} \quad (15)$$

where $\alpha > 0$, $x_0 > 0$. On the basis of Eq. (3) we find supersymmetric potentials-partners $V_1(x)$ and $V_2(x)$:

From Eq. (16) for $V_1(x)$ one can see that at the condition

$$\alpha = \frac{\square}{\sqrt{2m}} \quad (18)$$

potential $V_1(x)$ becomes constant. The penetrability coefficient relatively the propagation of the plane wave through this potential equal to one and, in this sense, the potential $V_1(x)$ is reflectionless. In accordance with Eq. (14), the penetrability coefficient of the potential $V_2(x)$ equals to one also:

$$|T_1|^2 = |T_2|^2 = 1. \quad (19)$$

Note the following property: the penetrability coefficient for the reflectionless potential is not changed with change of x_0 (at $x_0 > 0$).

At $x_0 < 0$ the region $x \in]-|x_0|, +|x_0|]$ appears, where the potentials have infinite high values and, in this sense, they have absolute opacity. A case $x_0 = 0$ is boundary.

$$V_1(x) = \begin{cases} \frac{\alpha}{(x-x_0)^{2n}} \left(\alpha - \frac{\square n |x-x_0|^{n-1}}{\sqrt{2m}} \right), & \text{at } x < 0, \\ \frac{\alpha}{(x+x_0)^{2n}} \left(\alpha - \frac{\square n |x+x_0|^{n-1}}{\sqrt{2m}} \right), & \text{at } x > 0, \end{cases} \quad (21)$$

$$V_2(x) = \begin{cases} \frac{\alpha}{(x-x_0)^{2n}} \left(\alpha + \frac{\square n |x-x_0|^{n-1}}{\sqrt{2m}} \right), & \text{at } x < 0, \\ \frac{\alpha}{(x+x_0)^{2n}} \left(\alpha + \frac{\square n |x+x_0|^{n-1}}{\sqrt{2m}} \right), & \text{at } x > 0, \end{cases} \quad (22)$$

Therefore, the potential $V_1(x)$ can be constant only when the condition from the following one is fulfilled:

$$n=0 \text{ or } n=1. \quad (23)$$

The condition $n=0$ gives trivial solutions. Let's consider another condition: $n=1$. In this case the potential $V_2(x)$ becomes reflectionless if the condition (18) is fulfilled. If the condition (23) is not fulfilled then one cannot reach the constancy of the potentials $V_1(x)$

$$\left(W(r) = \frac{\pm \alpha}{r+r_0} \right) \Rightarrow \begin{cases} V_1(r) = \frac{\pm \alpha}{(r+r_0)^2} \left(\alpha - \frac{\square}{\sqrt{2m}} \right), \\ V_2(x) = \frac{\pm \alpha}{(rx+r_0)^2} \left(\alpha + \frac{\square}{\sqrt{2m}} \right). \end{cases} \quad (24)$$

When the condition (18) is fulfilled then the potential $V_1(r)$ is constant, the potentials $V_1(r)$ and $V_2(r)$ are reflectionless, and scattering of a particle upon them is *resonant*. Note that a case $n=1$ is boundary between potentials with $n>1$ (where an incidence of particle upon a centre is possible) and the potentials with $n<1$ (where the incidence of the particle upon the centre is not possible).

4. ONE-DIMENSIONAL POTENTIAL

$V(x)=const / |x-x_0|^n$ AND SPHERICALLY SYMMETRIC POTENTIALS

$$V(x)=const / |r-r_0|^n$$

Now we consider more general case with the superpotential of the following form:

$$W(x) = \begin{cases} \frac{\alpha}{|x-x_0|^n}, & \text{at } x < 0, \\ \frac{\alpha}{|x+x_0|^n}, & \text{at } x > 0, \end{cases} \quad (20)$$

where $\alpha > 0$, $x_0 > 0$, n is a natural number. Find the potentials-partners $V_1(x)$ and $V_2(x)$:

or $V_2(x)$ by change of the coefficients α and m . If to change sign at $W(x)$, then the sign at the potentials $V_1(x)$ and $V_2(x)$ is changed also. Here, analysis described above remains applicable.

Now we generalise the analysis of the one-dimensional reflectionless potentials described above into spherically symmetric case (at $l=0$). Here, one need to use the functions $W(r)$ and $V_{1,2}(r)$ for the positive $r>0$ only. At $n=1$ we obtain:

5. CONCLUSIONS

On the basis of SUSY QM methods the condition is found, under which the potential, having the inverse power dependence on a space coordinate, becomes reflectionless for wave propagation through it. The potentials of such a type are interested in that they have enough obvious and simple form in a comparison on a number of potentials studying in [3,4], they are

expressed through elementary functions in an analytical form in a contradiction on a majority of shape invariant potentials studying in [5] and expressing with use of series, and they are considered often in problems of the scattering theory.

As further perspective, a problem of extension of a class of the reflectionless potentials on the basis of inverse power reflectionless potentials with use of canonical transformations of coordinates (this method is used for obtaining new exactly solvable potentials on the basis of known one and described in details in [5]) can be studied.

Here, we note that solving the equation

$$V_1(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2m}} \frac{dW(x)}{dx} = \text{const}, \quad (25)$$

one can find a general form of the function $W(x)$, which determines the reflectionless potentials. From here one can obtain all types of the reflectionless potentials. Here, partial solutions of Eq. (25) are:

$$\begin{aligned} W(x) &= \pm \frac{\alpha}{x - x_0}, \\ W(x) &= B \cdot \tanh(\alpha(x - x_0)), \\ W(x) &= \text{const}, \end{aligned} \quad (26)$$

where $B = \text{const}$. The superpotential $W(x) = B \tanh(\alpha(x - x_0))$ is known in literature (for example, see [5]).

REFERENCES

1. V.M. Chabanov, B.N. Zakhariev. Absolutely transparent multichannel systems. Unexpected peculiarities // *Physics Letters B*. 1993, v. 319, p. 13-15.
2. N. Saito, Y. Kayanuma. Resonant tunneling of a composite particle through a single potential barrier // *Journ. of Phys. Condens. Matter*. 1994, v. 6, p. 3759-3766.
3. B.N. Zakhariev, V.M. Chabanov. Qualitative theory of control of spectra, scattering, decays (Quantum intuition lessons) (Качественная теория управления спектрами, рассеянием, распадами (уроки квантовой интуиции)) // *Physics of elementary particles and atomic nuclei (ЭЧАЯ)*. 1994, v. 25, №6, p. 1561-1597 (in Russian).
4. B.N. Zakhariev, V.M. Chabanov. About the qualitative theory of elementary transformations of one- and multichannel quantum systems in the inverse problem approach (К качественной теории элементарных преобразований одно- и многоканальных квантовых систем в подходе обратной задачи) // *Physics of elementary particles and atomic nuclei (ЭЧАЯ)*. 1999, v. 30, №2, p. 277-320 (in Russian).
5. F. Cooper, A. Khare, U. Sukhatme. Super-symmetry and quantum mechanics // *Physics Reports*. 1995, v. 251, p. 267-385.
6. C.V. Sukumar. Supersymmetry and potentials with bound states at arbitrary energies: II // *Journ. of Phys. A.: Math. Gen.* 1987, v. 20, p. 2461-2481.

ОДНОМЕРНЫЕ ОБРАТНО СТЕПЕННЫЕ АБСОЛЮТНО ПРОЗРАЧНЫЕ ПОТЕНЦИАЛЫ

ТИПА $V(x)=\text{const} / |x - x_0|^n$

С.П. Майданюк

С помощью методов SUSY QM получено условие, при котором обратно степенной потенциал $V(x) = \alpha / |x - x_0|^n$ ($\alpha = \text{const}$, $x_0 = \text{const}$, $x \in]-\infty, +\infty[$, n – натуральное число) становится абсолютно прозрачным при прохождении через него плоской волны. Представлен анализ рассеяния частицы на сферически-симметричном потенциале $V(r) = \pm \alpha / |r - r_0|^n$ с учетом возможности абсолютной прозрачности.

ОДНОВИМІРНІ ОБЕРНЕНО СТЕПЕНЕВІ АБСОЛЮТНО ПРОЗОРІ ПОТЕНЦІАЛИ

ТИПУ $V(x)=\text{const} / |x - x_0|^n$

С.П. Майданюк

На основі методів SUSY QM отримано умову, при якій обернено степеневий потенціал $V(x) = \alpha / |x - x_0|^n$ ($\alpha = \text{const}$, $x_0 = \text{const}$, $x \in]-\infty, +\infty[$, n – натуральное число) стає абсолютно прозорим при проходженні крізь його плоскої хвилі. Представлено аналіз розсіювання частинки на сферично-симметричному потенціалі $V(r) = \pm \alpha / |r - r_0|^n$ з врахуванням можливості абсолютної прозорості.