A new type of chaotic billiards is introduced. Unlike the known ones, it contains scattering as well as focusing regions of the boundary and has no neutral components. The dumbbell-like from polymorphous billiards family is proposed. Its characteristic phase dynamics (at control parameter changes) is studied and Lyapunov exponent as well as invariant reflections density on the boundary are calculated. The chaotic behavior of the beams and their uniform stationary distribution are proved.

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1. INTRODUCTION

Billiards, i.e. systems with elastic or mirror reflections, occupy the central position in the deterministic chaos theory and have numerous physical applications. Chaotic billiards, in which the dynamics of all beams is chaotic, have gained a special popularity in deterministic chaos theory. In them, similar trajectories exponentially diverge in phase space, which causes stirring and ergodicity. Among chaotic billiards scattering Sinai billiards [1-3] and defocusing Bunimovich billiards [4-6] are distinguished. A distinctive geometric feature of scattering billiards is that all their boundary consists only of scattering components with negative curvature $K < 0$. The reflection from scattering components is always accompanied by angle widening of the bundle of initially close trajectories. This, eventually, leads to their chaotization. The boundary of defocusing billiards includes components with non-negative curvature $K \geq 0$. In these billiards, chaotization is assured by defocusing of the beams, which are first focused at reflection from convex components of billiard boundary. Defocusing mechanism is an alternative to scattering mechanism. Absolute chaos in billiards with only concave components ($K > 0$) is found only in case their absolute defocusing [6]. In smooth convex billiards this condition is not true. For defocusing mechanism, boundary singularities, such like return points, are needed. In stadiums with convex boundary, defocusing works due to the existence of rectilinear regions with zero curvature $K = 0$.

Chaotic billiards with mixed scattering character, i.e. billiards, whose boundary includes components of positive as well as of negative curvature, have never been considered before. It is obvious that beams scattering and (de)focusing are contrary by their nature. In general case, they will compete. So full chaos should be expected from billiards of only special type, in which these mechanisms are not in each other’s way, but supplement each other. Such chaotic billiards, whose boundary includes components with curvature of different sign, really exist. Among them are polymorphous billiards, introduced below.

2. POLYMORPHOUS AND DUMBBELL-LIKE BILLIARDS

Let us smoothly join (so that the tangent has no discontinuities) an even number of arcs taken from one circle to get a closed curve. We shall call the billiard limited by such a curve polymorphous billiards. Its boundary is formed by the arcs of the same circle and has everywhere constant curvature to sign. Some examples of such billiards one can see in Fig. 1.

Fig. 1. Geometry of polymorphous billiards: symmetric and asymmetric; 6(on the left) and 8 (on the right) orders

For closed boundary of polymorphous billiard it is necessary that initial number of circles arches should be even and not less than four. The distinctive feature of their boundary is that its curvature alternately changes.
its sign, i.e. $\text{sgn} K_j = (-1)^{|\alpha|}$, where (8). If radiuses of all the discs and circles which limit them are equal to one, then $K_j = (-1)^{|\alpha|}$. The curvature here is constant by its absolute value. The leap in the contact points of neighboring components affects only the vector of normal, unlike the everywhere continuous tangent field.

The simplest polymorphous billiard is a dumbbell-like billiard, the boundary of which is formed by arcs of four circles (Fig. 2). A smaller number of arcs is impossible, otherwise the smoothness of the boundary obtained would have been broken.

Fig. 2. Geometrical portrait of family of dumbbell-like billiards. It is shown the typical billiard trajectories

To study the “dumbbell” dynamics, we use geometric-dynamical approach [7-10], in which beams dynamics is described in a special symmetric phase space. Let us chose angle $\chi$ between the axis, connecting the centers of the convex components of the border and the beam, drawn to the point of contact between the convex and concave components as the control parameter of the dynamic system. This angle $\chi$ corresponds to the width of the middle of the dumbbell. It is changed from $\pi/2$ to $\pi/6$. At $\chi = \pi/2$ there is no narrow middle and we have a billiard in a circle instead of the dumbbell one. At $\chi = \pi/3$ the circles corresponding to the convex parts of dumbbell boundary, contact (inside of the billiard). At $\chi = \pi/2$ we have the most symmetric configuration. At $\chi = \pi/6$ the middle reaches its maximum and billiard falls into two ones.

3. CHAOTIC DYNAMICS OF “DUMBBELL”

The phase portrait gives us important information that billiard in a dumbbell has developed chaos. Fig. 3 shows changes in “dumbbell” phase portrait at changes of control parameter $\chi \in [\pi/6, \pi/2]$. At $\chi \in [0, \pi/6]$ the dumbbell falls into a pair of symmetric billiards in the form of “drops” with the identical phase portrait (Fig. 4).

Fig. 3. Phase portrait of family of dumbbell-like billiards in symmetric phase space at $\pi/6 < \chi < \pi/2$

At $\pi/6 < \chi < \pi/2$ there are two lacunas in the dumbbell phase space. So the billiard has $N_e$ topological type (sphere, stuck up with 6 Moebius loops). At $0 < \chi < \pi/6$ there is only one return lacuna. So topological type of “drop” is $N_d$. $\chi$ decreasing, the phase volume of “dumbbell” lacunas increases, but one of the “drop” lacuna decreases. Phase trajectories never get inside the lacunas. At intermediate $\chi = \pi/4$, lacunas overlap. Instead of a pair of isolated lacunas, in the symmetric phase space, there is one common region of classically illegal movement.

Fig. 4. Geometrical and phase portrait of family of dumbbell-like billiards at $0 < \chi < \pi/6$. By arrows it is noted the typical trajectories in symmetric phase space

Trajectories dynamics at $\chi \neq \pi/2$ is always chaotic. In the symmetric phase space of the dumbbell there are no traces of any elliptic (integrable) movement component. Regular trajectories like “whispering galleries” are ruined by lacunas, that appear at any “bottleneck”, how-
The phase portrait corresponds to the billiard dynamic in the asymptotic limit at arbitrarily great number of iterations. The dumbbell dynamics studies on finite time intervals shows that at very small \( \varepsilon \approx 10^{-3} \) (see Fig. 3 at the control parameter change from \( \chi = 1.57079 \) to \( \chi = 1.56879 \)) phase trajectories stay for a long time in intermediate rationally commensurable layers between the ruined invariant curves. For homogeneous filling of all the phase space, they need essentially larger number of iteration of the order \( 10^5 \) and more than at \( \varepsilon \approx 10^{-1} \). This lets us conclude that the chaoticization of the dumbbell starts by the scenario of forming cantori at ruining invariant curves near non-perturbed periodic orbits. Phase trajectories slowly ooze through these cantori. Invariant curves ruin the quickest near the periodic orbits with 2 period, joining the concave components of the dumbbell, i.e. near the phase points \((1/2, 3/2)\) and \((3/2, 1/2)\). This corresponds to more filled (dark) zones on the phase portrait. Let us note that the appearance of irregular invariant sets in the phase space of the dumbbell is connected with the breaks of involution and of its derivatives. The phase cascade rapidly multiples these breaks, so non-perturbed KAM-tori (invariant curves) of the circle billiard turn into the fractal cantori of the dumbbell. Deformation parameter \( \varepsilon \) increasing, apparent Cantor-structure collapses. It is changed by a structure of lacunas, connected with the geometric shadow for the billiard beams. At \( \varepsilon \approx 10^{-1} \) any phase trajectory fills the available phase space rather quickly during the time of the order about \( 10^5 \) iterations.

5. INVARIANT DISTRIBUTION

It is convenient to use the stationary reflection density, introduced in [12], for the description of statistic properties. In the symmetric phase space it is easier to calculate than full invariant measure. At small “bottleneck” of the dumbbell we shall decompose the billiard mapping by \( \varepsilon = \pi/2 - \chi \ll 1 \)

\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix} = B_{\varepsilon} \begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-1 & \alpha
\end{pmatrix} \begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix} + o(\varphi_1, \varphi_2)
\]

where \( \alpha = \alpha(\varepsilon) > 2 \) defines the average inclination of involution level; lines in the phase space. We shall limit ourselves to the main contribution and cast out the contributions to dynamics from the special set

\[
S = \bigcup_{n=0}^{\infty} B^{-n} (\varepsilon L)
\]

(the full preimage of the trajectories visiting the lacunas boundary) of the vanishing measure, \( \mu(S) = O(\varepsilon) \). The equation for one-point invariant density will have the following form

\[
\rho(\varphi) = \frac{1}{\alpha} \int \rho(\psi) \rho\left(\frac{\varphi + \psi}{\alpha}\right) d\psi + O(\varepsilon),
\]

Solving the equation (2) by the successive iterations method, taking into account that \( \alpha > 2 \) (this assures the convergence of series), we shall have

\[
\rho(\varphi) = 1 + O(\varepsilon)
\]

The solution (3) is coordinated with the normalization only under the condition that \( \rho(\varphi) = 1 \). For a chaotic billiard, due to its stirring, this is the only solution. The numeric calculation shows that stationary density \( \rho(\varphi) \) also stays constant at arbitrary \( \varepsilon \), see
Fig. 6. Stationary density of reflections on the billiard boundary. The full length of the billiard boundary is normalized to one.

So, in asymptotic limit, the beams visit the dumbbell boundary with approximately equal frequency. In the symmetric phase space, on the contrary, the phase points are distributed non-uniformly because of the lacunas presence.

6. CONCLUSIONS

A new type of polymorphous billiards with chaotic dynamics of the beams is proposed. Unlike the known chaotic billiards, their boundary includes scattering and focusing boundary components at the same time. So the chaoticization mechanism in such billiards is of mixed character. The reconstruction dynamics and phase reconstructions in one-parametric family of typical polymorphous dumbbell-like billiards are studied. The chaotic behavior of the beams and uniform distribution of the reflections are proved.

One should expect analogous behavior of the beams in polymorphous billiard of arbitrary form. The chaotic properties found let one use polymorphous billiard in applications. In particular, “dumbbell” form can be used for example, in atomic [13], microwave [14] or semiconductor billiards [15]. Chaos peculiarities in polymorphous billiards can also influence the character of light pass in optic nanoceramics microclusters, formed due to coagulating of ball-like nanoparticles etc.

REFERENCES