FREQUENCY ANALYSIS OF THE STOCHASTIC FILTERING USING TRANSFER FUNCTIONS. PART I: SINUSOIDAL INPUT

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Volterra-series-type analyses to communication systems with stochastic resonance driven by sine waves result to. The \( n \)-fold Fourier transform of the \( n \)-th Volterra kernel plays an important role in the analysis. Methods of computing transfer functions from the system equation are described and obtaining results are considered. It is shown, if the transfer functions are known, then the output signal can be obtained by substitution the transfer functions in general formulas derived from Volterra series representation. The obtained results showed that the amplitude of the sinusoid should not be greater than one to obtain a minor third harmonic. Besides, with an increase in the frequency of the sinusoid, the power of the output signal decreases sharply. Numerical calculations of the output signal driven by sinusoidal input were made to improve the accuracy and reliability of the obtained results. Comparative analysis showed the coincidence of the results of the calculation by different methods.

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INTRODUCTION IN VOLterra Series

Analysis

In communication systems often it is necessary to deal with the devices executing non-linear conversions. Volterra series are usually used for calculation of such devices. Wiener introduced Volterra series into non-linear circuit analysis [1].

The object of this paper is to present results of applying Volterra-series-type analyses to systems driven by sine waves.

Volterra series describe the output of a nonlinear system in degrees of input \( x(t) \). A substantial number of the communication system can be represented as Volterra series. The series for typical system can be writing as [2]

\[
y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\xi_1 ... \int_{-\infty}^{\infty} d\xi_n g_n (u_1, ..., u_n) \prod_{r=1}^{n} x(t - \xi_r),
\]

where \( y(t) \) is the output, \( x(t) \) – the input and the kernels \( g_n (u_1, ..., u_n) \) describe the system. The first-order kernel \( g_1 (u_1) \) is simply the familiar impulsive response of linear network. The higher order kernels of higher order impulse responses and serve to characterize the various orders of nonlinearity.

The coefficient \( 1/n! \) inserted A. Bedrosian and D. Rice [2] it because it simplifies many of equations.

The \( n \)-fold Fourier transform have the form [2, 3]

\[
G_n (f_1, ..., f_n) = \\
\int_{-\infty}^{\infty} d\xi_1 ... \int_{-\infty}^{\infty} d\xi_n g_n (u_1, ..., u_n) \exp \left[-j(f_1\xi_1 + ... + f_n\xi_n)\right] .
\]

(2)

\( G_0 \) is identically zero because our Volterra series starts with \( n = 1 \). \( G_1 (f_1) \) is the transfer function of linear network. For linear systems, the possible output frequencies are the same as the frequencies in the input. For non-linear systems, however, the relationship between the input and output frequencies is more complicated [4, 5].

Thus the transform of the \( n \)-th-order Volterra kernel is seen to be analogous to an \( n \)-th-order Volterra transfer function. In many cases \( G_n \) can be obtained without first computing \( g_n \).

The complete formulas are infinite series. Fortunately, in the study of communication system it is often possible to neglect terms of the Volterra series of order higher then the second or third. They are usually used because of fast increase in complexity [2, 3]. The \( n \)-fold Fourier transform considerably simplifies the solution of a large number of problems.

To calculate the transfer functions, we use the harmonic input method [2]. This method relies on the fact that a harmonic input must result in a harmonic output when (1) holds. System specified by the nonlinear differential equation considered [2, 3]

\[
F(d/dt)y + \sum_{l=2}^{\infty} a_l y^l = x(t) ,
\]

(3)

with the condition that system causally \( (y(t) \) vanish identically when \( x(t) \) does). It is assumed that one and only one such solution exists (it is proved in [3]) and the system is stable. \( F(d/dt) \) is a polynomial in \( d/dt \), and the coefficients in \( F(d/dt) \) and the coefficients \( a_l \) are independent of \( t, x, \) and \( y \).

The Volterra transfer functions for (3) can be written as [2]

\[
G_n (f_1, ..., f_n) = \frac{\sum_{l=0}^{\infty} a_l G_n^{(l)} (f_1, ..., f_n)}{F(j\omega_1 + ... + j\omega_l)} .
\]

(4)

The last equation is recurrence relation because \( G_n^{(l)} \) is given by

\[
G_n^{(l)} (f_1, ..., f_n) = l! \sum_{(\nu_1, \nu_2) \in N} a_{\nu_1} G_{\nu_1} (f_1, ..., f_{l+1}) \times
\]

\[
\times G_{\nu_2^{(2)}} (f_{l+1}, ..., f_{l+n}) ... G_{\nu_2^{(n)}} (f_n, ..., f_n) .
\]


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249
SINUSOIDAL INPUT FOR STOCHASTIC RESONANCE

Consider a general bistable dynamic system which can be described by the following stochastic differential equation [6, 7]

\[
dy = ay - by^3 + x(t),
\]

where \(a\) and \(b\) are positive, usually given in terms of system parameters,

\[
dx = \sqrt{2D} \delta(t) + \sqrt{2D} \eta(t),
\]

where \(D\) is the noise intensity. The input signal consists of the driving signal \(x(t) = s(t) = A \sin(2\pi f_0 t + \varphi)\) and the additive noise \(n(t)\) [6, 8]. This equation describes a stochastic resonance effect (SR) [6 - 8]. Volterra transfer function for \(x(t)\) are given in Table 1 for the general case (the equation 3) and for SR equation.

Table 1

<table>
<thead>
<tr>
<th>Volterra transfer function for eq. 3 [2]</th>
<th>Volterra transfer function for eq. 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ G_1 = \frac{1}{F(j\omega)} ]</td>
<td>[ G_2 = \frac{1}{-a + j\omega} ]</td>
</tr>
<tr>
<td>[ G_3 = \frac{2a_2 \sum G_1(f_1)G_2(f_2, f_3) + 6a_2G_1(f_1)G_2(f_2)G_1(f_3)}{F(j\omega_1 + j\omega_2 + j\omega_3)} ]</td>
<td>[ G_3 = \frac{-6b}{(-a + j\omega_1)(-a + j\omega_2)(-a + j\omega_3)} ]</td>
</tr>
</tbody>
</table>

When noise is absent, input signal can be writing as \(x(t) = s(t) = A \sin(2\pi f_0 t + \varphi)\). We will solve SR equation for this case. The leading terms output signal are given in Table 2.

Table 2

The output signal for eq. 3 [2]

| \[ e^{j\omega_0 t} \left( \frac{P^2}{4} G_2(f_p, -f_p) + \ldots \right) \] | \[ 0 \] |
| \[ + e^{-j\omega_0 t} \left( \frac{P^2}{4} G_2(f_p, f_p) + \ldots \right) \] | \[ A \left( \frac{\omega_0^2 + \omega_0^2}{16} \right)^2 \cos \omega_0 t + \right. |
| \[ + e^{j3\omega_0 t} \left[ \frac{P^3}{8} G_3(f_p, f_p, f_p) + \ldots \right] \] | \[ + \frac{A}{(a^2 + \omega_0^2)^2} \left( \omega_0^2 + \frac{3a^2b \omega_0^2}{2(a^2 + \omega_0^2)^2} \right) \sin \omega_0 t \] |

We will define the output signal of non-linear system by the Runge-Kutta method and by Volterra series. Results of calculations are given in Figure for \(A = 1, f = 0.5\) Hz.

Results from the Figure show, that in the numerical calculation by Runge-Kutta method takes place transient that lasts about two periods of a signal. Further the results of calculation of an output signal received by both methods match.

We will determine the power of an output signal. Powers of the first and third harmonica are specified in Table 3.
The power of the output signal for eq. 3 [2]  

\[ S_{\text{eb}} \approx \frac{A^2}{2} G_1(f_0) + \frac{A^4}{16} G_3(f_0, f_0, -f_0) \]

The power of the output signal for eq. 4  

\[ S_{\text{eb}} \approx \frac{a^2 A^2}{4(a^2 + \omega_0^2)^2} - \frac{3 A^4 b a}{8(a^2 + \omega_0^2)^3} + \frac{9 A^6 b^2}{64(a^2 + \omega_0^2)^4} + \frac{A^2 \omega_0^2}{4(a^2 + \omega_0^2)^2} \]

\[ S_{\text{eb}} \approx \frac{p^3}{48} G_3(f_0, f_0, f_0) \]

\[ \frac{A^4 b^2}{64(a^2 + \omega_0^2)^3(a^2 + 9\omega_0^2)} \]

Results of Table 3 show that, with an increase in the frequency of the sinusoid, the power of the output signal decreases drastically.

\[
S_{\text{eb}} / S_{\text{eb}} = \frac{a^2 16(a^2 + \omega_0^2)(a^2 + 9\omega_0^2) - 24(a^2 + 9\omega_0^2)}{A^4 b^2} + \frac{9(a^2 + 9\omega_0^2)}{(a^2 + \omega_0^2)} + \frac{16\omega_0^2(a^2 + \omega_0^2)(a^2 + 9\omega_0^2)}{A^4 b^2}.
\]

We can calculate the powers relation of the first and third harmonicas as:

\[ S_{\text{eb}} / S_{\text{eb}} = \frac{a^2 16(a^2 + \omega_0^2)(a^2 + 9\omega_0^2) - 24(a^2 + 9\omega_0^2)}{A^4 b^2} + \frac{9(a^2 + 9\omega_0^2)}{(a^2 + \omega_0^2)} + \frac{16\omega_0^2(a^2 + \omega_0^2)(a^2 + 9\omega_0^2)}{A^4 b^2}.
\]

The obtained results showed that the amplitude of the sinusoid \( A \) should not be greater than one to obtain a minor third harmonic.

**CONCLUSIONS**

The Volterra series is a powerful tool that can be used to describe a wide class of non-linear systems.

Results of applying Volterra series analysis to systems with SR effect driven by harmonic input showed what output signal contains the 1st and 3rd harmonics, what output signal contains the 1st and 3rd harmonics, the power of the output signal decreases sharply and with an increase in the frequency of the sinusoid, the power of the output signal decreases sharply.

Results of calculations showed that numerical calculation and calculation by Volterra series match. At the same time numerical calculation is followed by transient which lasts about two periods of oscillations. Transient take place in radioengineering devices [9].

In the present paper the method of transfer functions is developed. The received gear Volterra transfer function for the systems with SR effect will allow to receive further expressions for an output signal in case of input white Gaussian noise and an input additive mix of a harmonic signal and white Gaussian noise.

**REFERENCES**


**ЧАСТОТНЫЙ АНАЛИЗ СТОХАСТИЧЕСКОЙ ФИЛЬТРАЦИИ С ПОМОЩЬЮ ПЕРЕДАТОЧНЫХ ФУНКЦИЙ**.

**ЧАСТЬ I: СИМУСОИДАЛЬНЫЙ ВХОДНОЙ СИГНАЛ**

О.Ю. Харченко, Ю.Ф. Лонин, А.Г. Пономарев

Приведен анализ на основе рядов Вольтерра применительно к телекоммуникационным системам, обладающим эффектом статистического резонанса, возбуждаемым синусоидальным сигналом. Важную роль при таком анализе играет многомерное преобразование Фурье. Описаны способы вычисления передаточных функций, исходя из уравнений системы, и рассмотрены результаты расчетов. Сравнительный анализ численных расчетов и расчетов на основе рядов Вольтерра показал совпадение результатов расчетов.

**ЧАСТОТНЫЙ АНАЛИЗ СТОХАСТИЧЕСКОЙ ФИЛЬТРАЦИИ ЗА ДОПОМОГОЮ ПЕРЕДАТОЧНИХ ФУНКЦІЙ. ЧАСТИНА І: СИМУСОІДАЛЬНИЙ ВХІДНИЙ СІGNАЛ**

О.Ю. Харченко, Ю.Ф. Лонін, А.Г. Пономарев

Наведено аналіз на основі рядів Вольтерра стосовно телекомунікаційних систем, що володіють ефектом статистично- го резонансу, які збуджуються синусоїдальним сигналом. Важливу роль при такому аналізі грає багатовимірне перетворення Фур’є. Описано способи обчислення передаточных функцій, виходячи з рівнянь системи, і розглянуто результати розрахунків. Порівняльний аналіз чисельних розрахунків і розрахунків на основі рядів Вольтерра показав збіг результатів розрахунку.

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