OSCILLATORY INSTABILITY

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The dynamics of the oscillator system is investigated. The conditions under which this dynamics becomes unstable are determined. In particular, it is shown that plasma in constant magnetic field becomes unstable if its density exceeds a certain critical value. In this case, instability develope (oscillatory instability). It is shown that random dynamics of the particles suppresses oscillatory instability.

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INTRODUCTION

Plasma, located in an external magnetic field, is a key component in the program of controlled thermonuclear fusion. The features of its confinement in various magnetic configurations are studied in detail. Numerous hydrodynamic and kinetic instabilities have been investigated (see, for example, [1 - 3]). It has been shown [4 - 6] that at plasma densities > 10^{14} cm^{-3} there are problems with its retention. This phenomenon has even acquired its own name "density limit". A phenomenological criterion for the plasma density limit, the Greenwald limit, is obtained. However, the physical mechanisms responsible for existence of restrictions on the density of confined plasma are still unknown. For years there is a search for ways to overcome this limit. In [7], as an experimental success, is reported that at the Alcator C-Mod tokomak a plasma density of 1.5×10^{14} cm^{-3} was obtained, i.e. one and a half times the calculated limit.

It is possible that the mechanism described in this paper will to some extent allow a deeper understanding of the cause of the "density limit". Below, in the second and third sections, we consider systems of coupled oscillators. The conditions for their instability are determined. In the fourth section shown that the plasma, which is in a magnetic field, can stably exist if its density does not exceed a certain critical value. 'Extra' particles ejected from the ensemble. The fifth section is devoted to the description of a possible mechanism for the suppression of oscillatory instability. In conclusion, the main results are formulated.

1. DYNAMICS OF ENSEMBLE OF LINEAR OSCILLATORS

Suppose we have a system with Hamiltonian:

$$H = \sum_{i=0}^{N}\left(\frac{p_i^2}{2} + \omega_0^2\frac{q_i^2}{2}\right) + \mu\cdot q_0\cdot\sum_{j=1}^{N} q_j . \quad (1)$$

This system is a system of coupled linear oscillators. Hamiltonian (1) corresponds to the system of equations for describing the dynamics of coupled linear oscillators:

$$\ddot{q}_i + \omega_0^2 q_i = -\mu\cdot q_0 .$$
$$\ddot{q}_0 + \omega_0^2 q_0 = -\mu\cdot\sum_{i=1}^{N} q_i . \quad (2)$$

For simplicity, we consider a system in which all oscillators are connected with each other only through a zero oscillator (Fig. 1). The normal frequencies of such a system are easy to find. To do this, we will look for the solution of system (2) in the form:

$$q_i = a_i \exp\left(i\cdot\omega\cdot t\right),\ a_i = const . \quad (3)$$

Substituting this solution into (2), we obtain the dispersion equation:

$$\left(-\omega^2 + \omega_0^2\right)^2 = \mu^2 N. \quad (4)$$

Equation (4) gives the following expressions for normal frequencies:

$$\omega = \pm\omega_0\sqrt{1 \pm \mu\cdot\sqrt{N}/\omega_0^2} . \quad (5)$$

The signs "+" and "-" in the formula (5) before the root and under the root are independent. It can be seen that even with a very small coupling coefficient, but with a large number of oscillators, one of the normal frequencies can be very small (for the case of the sign "-" under the root). If the inequality holds:

$$\mu\cdot\sqrt{N} > \omega_0^2 , \quad (6)$$

then such an ensemble cannot exist. It collapses. A numerical analysis of the dynamics of the system (2) fully confirms this result. So, for example, if ten oscillators at the initial time placed randomly in the vicinity of the bottom of the potential well and the coupling coefficient is less than 0.3, then oscillations of the oscillators are limited. However, if we slightly increase the coupling coefficient ($\mu = 0.3334$), then the dynamics become unstable. Destruction criterion (6) is fulfilled. The ensemble collapses (Fig. 2).

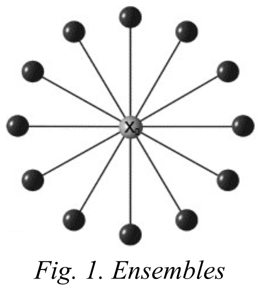

Fig. 1. Ensembles of oscillators. Connection occur only through central oscillator

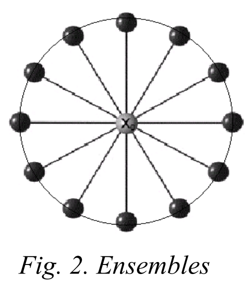

Fig. 2. Ensembles of oscillators. Connection occur through the central oscillator and between nearest neighbors

The ensemble considered above is a simplest model. The more realistic model is the ensemble in which all oscillators are connected with the central oscillator, and also coupled with their nearest neighbors. In addition, the frequency of the central oscillator is different from the frequency of the other oscillators. The system of equations that describes the dynamics of such an ensemble has the following form:

$$\ddot{q}_i + \omega_0^2 q_i = -\mu \cdot q_0 - \mu_1 (q_{i+1} + q_{i-1}), \quad (7)$$
$$\ddot{q}_0 + \omega_1^2 q_0 = -\mu \cdot \sum_{i=1}^{N} q_i \ .$$

To find the conditions for the existence of such an ensemble, it is convenient to rewrite the system (7) in the form:

$$\ddot{Q} + \omega^2 Q = -\mu \cdot N \cdot q_0, \quad (8)$$
$$\ddot{q}_0 + \omega_1^2 q_0 = -\mu \cdot Q \ ,$$

where $\omega^2 = \omega_0^2 + 2\mu$, $Q = \sum_{i=1}^{N} q_i$.

We seek solutions system (8) in the form: $Q \sim \exp(i \cdot \omega \cdot t)$. Then for normal frequencies the following expression can be obtained:

$$\omega^2 = \frac{1}{2}\left(\omega_1^2 + \omega_2^2\right)\left[1 \pm \sqrt{1 - 4\frac{\left(\omega_1^2 \omega_2^2 - \mu^2 N\right)}{\left(\omega_1^2 + \omega_2^2\right)^2}}\right], \quad (9)$$

with $\omega_2^2 = \omega_0^2 + 2\mu$.

All the features of the dynamics of such system are similar to the features of the previous system of oscillators. The condition for the destruction of this system is:

$$\mu^2 N > \omega_2^2 \omega_1^2 \ . \quad (10)$$

2. ENSEMBLE OF NONLINEAR OSCILLATORS

Let's write the system of differential equations, which describes the dynamics of an ensemble of mathematical pendulums, in which each oscillator is connected with any other:

$$\ddot{x}_i + \sin x_i = -\mu \cdot \sum_{j \neq i}^{N} x_j \quad i, j \in \{0, 1, 2....N\}. \quad (11)$$

The system of equation (11) was investigated by numerical methods. The coupling coefficient between the oscillators was chosen small ($\mu < 10^{-3}$). The particles, at the initial time were located in the vicinity of the equilibrium state. In the general case, during certain time interval, the particles oscillate in the potential having been captured by this potential. However, depending on the magnitude of the coupling coefficient, the initial position of the particles in the potential and on their number, some of the particles are ejected from the potential. Departure of one of the particles is shown in the fig. 3. This case corresponds to the dynamics of ten oscillators that are connected by the coupling coefficient $\mu = 0.0005$. The maximum dimensionless initial velocity of one of the particles (not necessarily theone ejected) was equal to 0.4. It should be noted that the time of particle departure from the potential is very sensitive to small changes in the potential itself, the coupling coefficients, and particle position. To this one can add that the dynamics of all particles is locally unstable [9, 10]. The dynamic chaos is developing. If one removes the particles that flew out of the ensemble, then the dynamics of the remaining particles remain restricted.

3. DYNAMICS OF PLASMA PARTICLES IN A MAGNETIC FIELD

Consider the motion of particles with a charge e in an external magnetic field directed along the axis $z : \vec{H}_0 = \{0, 0, H\}$. The particles rotate around the magnetic field lines, thus move with acceleration and radiate. We consider the ideal plasma; therefore, the Coulomb interaction of particles not be taken into account.

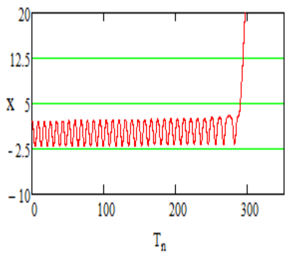

Fig. 3. The representative dynamics of particles that leave an ensemble of interacting particles. At that, this dynamics is characteristic of both linear and nonlinear oscillators

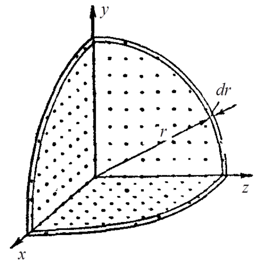

Fig. 4. The integration domain in the formula (19). R_{max} maximum distance at which oscillators can interact

Let's assume that the interaction is carried out using only fields that particles emit during rotation. The electric field strength in the vicinity of another particle will be:

$$\vec{E} = -\frac{e \cdot \dot{\vec{v}}}{c^2 R} \ . \quad (12)$$

Here e – is the charge of the particle; $\dot{\vec{v}}$ – is the acceleration of the particle as it rotates; c – is the velocity of light; R – is the distance between particles.

The interaction of two particles will be examined first with the motion transverse relative to the magnetic field. The dynamics of one particle can be described by the following equation:

$$m\ddot{\vec{r}} = \frac{e}{c}\left[\vec{v}\vec{H}_0\right] - \frac{e^2 \dot{\vec{v}}}{c^2 R} \ . \quad (13)$$

This vector equation is convenient to rewrite in the form of a system of equations for the velocity components of the first particle:

$$\dot{v}_{x1} = \omega_H v_{y1} - \mu \cdot \dot{v}_{x2}, \quad (14)$$
$$\dot{v}_{y1} = -\omega_H \cdot v_{x1} - \mu \cdot \dot{v}_{y2} \ ,$$

where $\omega_H = eH / mc$ is the cyclotron frequency of rotation of the particle in magnetic field, and $\mu = e^2 / Rmc^2$ is the influence of the second particle on the dynamics of the first particle (coupling coefficient between particles).

Differentiating the system of equations (14) and taking into account the system (14) itself, one can obtain the following system of equations:

$$(\ddot{v}_{x1} + \omega_H^2 \cdot v_{x1}) = 2\mu\omega_H^2 v_{x2},$$
$$\ddot{v}_{y1} + \omega_H^2 v_{y1} = 2\mu\omega_H^2 v_{y2}. \quad (15)$$

Similar systems of equations can be written for the dynamics of the second particle. If there are many particles, then the equations that describe the dynamics of the velocity components can be represented as:

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$$\ddot{v}_k + v_k = 2\sum_{j\neq k}\mu_j v_j. \quad (16)$$

The dependent variable v_k in equation (16) defines either x or the y component of the velocity of the k particle. In addition, there has been introduced a new time $\tau = \omega_H t$. The coefficients of the connection μ_j, which stand in the right side of equation (16) under the sign of the sum, differ from each other only by the distance between the particles R_j. Note that with a large number of oscillators ($N >> 1$) and small coupling coefficients ($\mu_j << 1$), the right side of equations (16) is the same for all oscillators (for all k). In such a case, equations (16) can be significantly simplified:

$$\ddot{v}_k + \left(1 - 2\sum_{j=1}^{N}\mu_j\right)v_k = 0. \quad (17)$$

Then the condition of instability has the form:

$$f(N) \equiv 2\sum_{j=1}^{N}\mu_j > 1. \quad (18)$$

The condition (18) can be rewritten for particle density. Indeed, the total number of interacting particles is equal to the density of particles multiplied by the volume occupied by the interacting particles ($N = nV$). The volume of a spherical layer of radius r and thickness dr is equal $V_{laer} = 4\pi r^2 dr$. Using this, the left side of inequality (18) can be rewritten (Fig. 4):

$$2\sum_{j=1}^{N}\mu_j = 8\pi n \int_0^{R_{\max}} \frac{r^2 dr}{r}\frac{e^2}{mc^2} = 4\pi \frac{e^2}{mc^2} n R_{\max}^2. \quad (19)$$

And the condition (17) take the form:

$$n > 3\cdot 10^{11} / R_{\max}^2. \quad (20)$$

4. SUPPRESSION OF THE OSCILLATORY INSTABILITY

There is question about possible mechanisms for the suppression of oscillatory instability. Taking into account the mechanism of instability, it can be expected that the occurrence of chaotic particle motion will suppress this instability. Below it is shown that such instability suppression mechanism exists. To prove this, in the equation (17) instead of the function $f(N) = 2\sum_{j=1}^{N}\mu_j$ that depends only on the number of interacting oscillators (N), we introduce an elementary random function: $\varepsilon(t) = V(t) f(N)$. To simplify the analysis, assume that this new function has the following properties:

$$\langle\varepsilon(t)\rangle = 0, \quad \langle\varepsilon(t)\varepsilon(t_1)\rangle = B(t)\delta(t - t_1). \quad (21)$$

For further analysis, it is convenient to rewrite the equation (17) in canonical form:

$$\dot{v} = a, \quad \dot{a} = -\left[1 - \varepsilon(t)\right]v. \quad (22)$$

Let's write the equations describing the dynamics of the first two moments:

$$\langle\dot{v}\rangle = \langle a\rangle \; \langle\dot{a}\rangle = -\langle v\rangle + \langle\varepsilon(t)v\rangle. \quad (23)$$

This system is for the first moments. The system for the second moments will be:

$$\frac{1}{2}\frac{d}{d\tau}\langle v^2\rangle = \langle va\rangle, \quad \frac{1}{2}\frac{d}{d\tau}\langle a^2\rangle = -\langle va\rangle + \langle\varepsilon va\rangle,$$
$$\frac{d}{d\tau}\langle va\rangle = \langle a^2\rangle - \langle v^2\rangle + \langle\varepsilon v^2\rangle. \quad (24)$$

The systems of equations (23) and (24) have the items which have correlation: $\langle\varepsilon(t)R[\varepsilon]\rangle$. For their decoupling, it is convenient to use the Furutsu-Novikov formula (see, for example, [8]):

$$\langle\varepsilon(t_1)R[\varepsilon]\rangle = B(t_1)\left\langle\frac{\delta R[\varepsilon]}{\delta\varepsilon(t_1)}\right\rangle, 0 < t_1 \le t. \quad (25)$$

here $\frac{\delta R[\varepsilon]}{\delta\varepsilon(t_1)}$ is the variation (functional) derivative.

The values of functional derivatives we need will be found from equations (23) and (24):

$$\frac{\delta a}{\delta\varepsilon} = v, \quad \frac{\delta v}{\delta\varepsilon} = 0, \quad \frac{\delta v^2}{\delta\varepsilon} = 0,$$
$$\frac{\delta a^2}{\delta\varepsilon} = 2va, \quad \frac{\delta av}{\delta\varepsilon} = v^2. \quad (26)$$

Using these formulas and formula (25), we finally get a system of equations for describing the dynamics of the first and second moments.

For the first moments:

$$\langle\dot{v}\rangle = \langle a\rangle, \quad \langle\dot{a}\rangle = -\langle v\rangle, \text{ or } \langle\ddot{v}\rangle + \langle v\rangle = 0. \quad (27)$$

For the second moments:

$$\frac{1}{2}\frac{d}{d\tau}\langle v^2\rangle = \langle va\rangle, \; \frac{1}{2}\frac{d}{d\tau}\langle a^2\rangle = -\langle va\rangle + B\langle v^2\rangle,$$
$$\frac{1}{2}\frac{d}{d\tau}\langle va\rangle = \langle a^2\rangle. \quad (28)$$

From the system of equations (27) it follows that the dynamics of the first moments does not differ from the dynamics of the original oscillators. They continue to oscillate with a cyclotron frequency (ω_H). The dynamics of the second moments turns out to be unstable. Indeed, let us look for the solution of system (28) in the form $\sim \exp(\lambda\tau)$. Then, to find the value λ, we obtain the following algebraic equation:

$$\lambda\left(\lambda^2 + 2\right) = 4B. \quad (29)$$

For our purposes, the solution of equation (29) is sufficient to give in two limiting cases:

$$\text{if} |\lambda|^2 << 2, \text{ then } \lambda = 4B, \quad (30)$$

$$\text{if} |\lambda|^2 >> 2, \text{ then } \lambda = (4B)^{1/3}. \quad (31)$$

The real parts of the other two roots are negative.

Above (formula (18)) we have seen that oscillatory instability develops when $f(N) \ge 1$. It can be seen from formula (30) that the fluctuation instability develop at much lower plasma densities. The increment of fluctuation instability at low plasma densities ($f(N) << 1$) increases in proportion to the square of the density. At high densities ($f(N) >> 1$) (as follows from formula (31)), its growth is weakened (as follows from formula (31)), its growth is weakened.

Thus, the presence of random dynamics of the particles disrupts oscillatory instability. The nature of the dynamics of the average particle velocities is maintained oscillatory. They oscillate with cyclotron frequency (ω_H). The energy of these oscillations is increasing. The scattering of particles in velocities is increasing. The growth of oscillator energy occurs due to fluctuation energy ($V(\tau)$). It should be noted, however, that the considered scenario of the suppression of oscillatory instability can be realized only in the case when the random dynamics of the particles already exist. Estimates of dynamic chaos regimes show that they are developing too slowly to suppress instability. An alternative is to impact on the plasma by intense noise fields.

5. DISCUSSION AND CONCLUSIONS

Let's formulate the most important results:

1. An ensemble of charged particles (electrons) in magnetic field can stably exist only if the density of these particles is less than a certain critical number that can be found from formula (20)). The "extra" particles are removed from the ensemble (see sections 2 and 3). Note that the instability increment can be abnormally large. The development of this instability (oscillator instability) means that plasma, for example a plasma cylinder, at densities above 10^{11} begins to dress with a coat of high-energy electrons. The plasma rod will be positively charged. As result there appear holding potential.

Note that if the plasma density exceeds a critical value, then it is apparently impossible to suppress the oscillatory instability. However, if the plasma density increases gradually and there is a random component in the particle dynamics, the dangerous oscillatory instability can be suppressed. In this case such instability replaced by fluctuation instability (the fifth section). This instability can be useful for both plasma confinement and plasma heating.

2. Above, the main attention was paid to the conditions for the destruction of the oscillatory dynamics of an oscillator system. However, stable ensembles with a large number of oscillators can also have considerable interest. We draw attention to some features of the detected evolution of particle dynamics depending on plasma density changes. The characteristic frequency of the particle velocity oscillation decreases with the increasing particle density

$$(v \sim v_T \cos(\Omega\tau) ; \Omega = \sqrt{1-2\sum_{j=1}^{N}\mu_j}).$$

And the corresponding particle displacement increases ($r \sim v/\Omega$). Such increasing in particle deflection may be undesirable.

3. Note that the well-known models of systems consisting of a large number of oscillators (e. g., in [11-13]), formulate in one form or another conditions under which the dynamics of an ensemble of coupled oscillators remain oscillatory. Such an approach is natural, since unstable ensembles do not exist. They fall apart. Therefore, only the properties of oscillatory ensembles are studied. As a result, the question of the development of instabilities in a system of a large number of coupled oscillators has been little studied

4. The models described above are as simple as possible. In the case, when under experimental conditions, for example, the magnetic field is inhomogeneous, the partial frequencies of the oscillators become different. The efficiency of the interaction of oscillators with different frequencies is much less than the efficiency of the interaction of identical oscillators. Therefore, it can be expected that the required number of particles for the destruction of an ensemble will be greater than, for example, formula (20) determines.

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ОСЦИЛЛЯТОРНАЯ НЕУСТОЙЧИВОСТЬ

В.А. Буц

Исследована динамика осцилляторной системы. Определены условия, при которых эта динамика становится нестабильной. В частности, показано, что плазма в постоянном магнитном поле становится нестабильной, если ее плотность превышает некоторое критическое значение. В этом случае развивается нестабильность (колебательная неустойчивость). Показано, что случайная динамика частиц подавляет колебательную неустойчивость.

ОСЦИЛЯТОРНА НЕСТІЙКІСТЬ

В.О. Буц

Досліджено динаміку осциляторної системи. Визначені умови, при яких ця динаміка стає нестабільною. Зокрема, показано, що плазма в постійному магнітному полі стає нестабільною, якщо її густина перевищує деяке критичне значення. У цьому випадку розвивається осциляторна нестабільність (коливна нестійкість). Показано, що випадкова динаміка частинок пригнічує осциляторну нестійкість.