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MODIFIED STEFAN CONDITION IN STEFAN PROBLEM

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The two-phase one-dimensional Stefan problem (SP) with the boundary between the phases moving with time is considered. The position of the boundary is determined by the modified Stefan condition (MSC), which is obtained from the original nonlinear diffusion equation by integrating over a thin transition layer, and by tending its thickness to zero. Upon receipt of the MSC, the diffusion coefficient is represented by the sum of the Heaviside step functions. It is shown that the MSC differs from the standard one in that in the latter, the derivatives of the concentrations with respect to the phase coordinates are interchanged. An expression for the displacement of the interphase boundary is obtained, which, as in the standard SP, is proportional to the square root of time. The results of using the MSC are confirmed by experimental data on the displacement of the Cu/Sn interface during diffusion bonding during isothermal annealing.

INTRODUCTION

The Stefan problem (SP) appeared as a result of research into the melting of the Earth's polar ice cap [1, 2]. From a mathematical point of view, SP is a boundary value problem for a second-order partial differential equation of parabolic type [3], which describes the movement of the interface between two phases over time. The interface between the phases moves as a result of external thermal action, accompanied by a change in the phase states of the substance (dissolution or crystallization). The speed of movement of the interfacial boundary (IB) is set by the difference in diffusion flows from one side and the other.

In addition to the problem of ice melting with a moving boundary between water and ice, examples of physical processes with a moving interface are, for example, problems of melting a solid with an unknown boundary between the solid and liquid phases, problems of concentration redistribution as a result of mutual diffusion in metal alloys with a moving boundary phase separation of different chemical composition, the problem of solidification, metal and non-metal castings [4].

For the first time, the study of the motion of the phase boundary of a homogeneous liquid-solid body was carried out in [5] by French scientists, corresponding members of the St. Petersburg Academy of Sciences G. Lame and B.P. Clapeyron. This formulation of the problem is due to the search for physical models that describe the formation of the earth's crust. In this work, it was found that the thickness of the solid phase formed during the cooling of a homogeneous liquid increases with time in proportion to $\sqrt{t}$, where t is the process development time.

The work [6] considers some issues of thermal conductivity in a melting ice prism and, as an application, the temperature distribution in a rod with different thermal constants at negative and positive temperatures. In this paper, it is shown that, as in the Lame and Claiperon problem, the temperature difference boundary shifts proportionally to $\sqrt{t}$.

Since the first publication on the study of the motion of the phase boundary, a large number of works have been done on this topic, which later became known as the SP. They are devoted to the solution of the SP with a moving and free boundary (M-FBP) for the diffusion equation (H-DE) both in theoretical and applied consideration [7].

In this paper, as an example, we will focus on the analysis of one of the SP with a moving and free boundary (M-FBP) for the diffusion equation (H-DE).

Diffusion phase transformation in solids is often modeled in terms of the two-phase Stefan model [8]. In this case, the mathematical formulation of the problem takes the form of two partial differential equations (with appropriate boundary conditions) describing diffusion in both phases, and a material balance equation at the interface. The problem of the two-phase Stefan model is solved by analytical methods, provided that the diffusion coefficients are constant, and the boundary value and initial conditions remain unchanged.

In this paper, we propose an analytical method for solving a one-dimensional two-phase SP diffusion interaction in a binary metallic system, in which a new method for formulating the boundary condition on a moving boundary is proposed.

ONE-DIMENSIONAL TWO-PHASE SP IN A SEMI-BOUNDED DOMAIN WITH A MODIFIED MOVABLE INTERPHASE BOUNDARY

Let us consider the process of diffusion interaction in a binary metallic system A–B with phases $i = \alpha, \beta$, which are regular solid solutions. On the schematic representation of the geometry of diffusion interaction in the (N, x) plane, we denote by $s(t)$ position of the moving interface (Fig. 1), where $N(x, t)$ is the concentration of the binary two-phase system, x is the coordinate, $t > 0$ – time. On Fig. 1 α – the phase is located in the region $0 \le x < s(t)$, and β – the phase is located in the interval $s(t) < x < \infty$, $N^i(x, t)$ is the concentration of the i phase. For simplicity of calculations, we assume that far from the interface, the concentration of phases is constant, i.e. $N^\alpha(0 \le x <$

$s(t), t) = N^{\alpha}(0,t) = N_{\alpha}, N^{\beta}(s(t) < x < \infty, t) = N^{\alpha}(\infty, t) = N_{\beta}$.

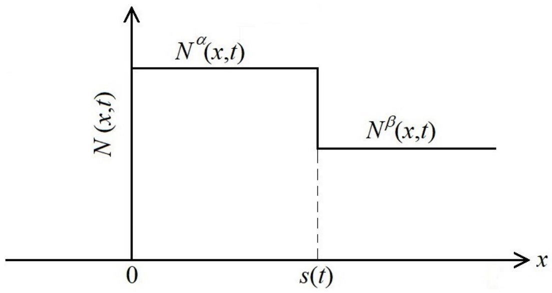

Fig. 1. Scheme of distribution of concentrations in a binary metal system

The equation for changing the phase concentration of a one-dimensional two-phase SP is written in a generalized form:

$$\frac{\partial}{\partial t}N(x,t) = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial}{\partial x}N(x,t)\right), \quad (1)$$

where $D(x,t)$ – coordinate-dependent diffusion coefficient.

As conditions on the boundaries of the region, we set the following:

$$N(0,t) = N_{\alpha}; \quad (2)$$

$$N(\infty,t) = N_{\beta}; \quad (3)$$

$$N(s(t),t) = N_s. \quad (4)$$

where $N_{\alpha} > N_s > N_{\beta}$.

Equation (1) should be supplemented by the equation of motion of the interface. In the literature, this equation is known as the Stefan condition [1]. From physical considerations, it is clear that the motion of the boundary occurs when there is a difference in particle fluxes from one phase to another. Therefore, the driving force of the displacement of the interface is contained in the initial conditions (2). In the classical formulation of SP, to describe the diffusion of heat between water and ice, this condition has the form:

$$\left.\begin{array}{c} T_2(x) = T_1(x) = 0 \\ K_2\dfrac{\partial T_2}{\partial x} - K_1\dfrac{\partial T_1}{\partial x} = L\rho\dfrac{\partial s(t)}{\partial t} \end{array}\right\} if \;\; x = s(t), \quad (5)$$

where K_i – thermal conductivity coefficients of phases; $T_i(x)$ – phase temperature; ρ – phase density; L – latent heat needed to melt ice.

The last of the conditions (5) was obtained based on the balance of the amount of heat supplied to the interface from the water and the amount of heat lost to ice, provided there are no heat sources at the interface.

Let us find solutions to equation (1) with boundary conditions (2)–(4). Since the diffusion coefficients $D(x,t)$ far from the interface are constants, solutions (1) in these regions can be represented as [9]:

$$N^{\alpha}(x,t) = N_1^{\alpha} - N_2^{\alpha}\mathrm{erf}\left(\frac{x}{2\sqrt{tD^{\alpha}}}\right),$$
$$N^{\beta}(x,t) = N_1^{\beta} - N_2^{\beta}\mathrm{erf}\left(\frac{x}{2\sqrt{tD^{\beta}}}\right), \quad (6)$$

where $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}d\xi$ – error function, N_1^{α}, N_2^{α}, N_1^{β}, N_2^{β} – constants determined from conditions (2)–(4).

From the value of concentrations at the boundaries of the region (2), (3), we determine the constants N_1^{α}, N_2^{α}:

$$N_1^{\alpha} = N_{\alpha}, N_1^{\beta} = N_{\beta}. \quad (7)$$

With constants $N_2^{\alpha}, N_2^{\beta}$ determine the conditions at the interface (4):

$$N_2^{\alpha} = \frac{N_{\alpha} - N_s}{\mathrm{erf}\left(\frac{s(t)}{2\sqrt{tD^{\alpha}}}\right)},$$
$$N_2^{\beta} = \frac{N_{\beta} - N_s}{\mathrm{erf}\left(\frac{s(t)}{2\sqrt{tD^{\beta}}}\right)}. \quad (8)$$

From (8) implicitly follows the equality $s(t) = A\sqrt{t}$, where A – constant. Thus, from (8) we obtain the well-known result of the SP on the dependence of the displacement of the interface on time [1, 9].

MODIFIED STEPHAN CONDITION AT THE INTERPHASE BOUNDARY

In this article, we propose a new, modified, Stefan condition in the SP. This condition can be obtained from equation (1) by integrating over a thin transition layer with a thickness $\varepsilon \ll l$ on both sides of the phase boundary: $s(t) - \varepsilon \leq x \leq s(t) + \varepsilon$. After the integration operation, the layer thickness tends to zero ($\varepsilon \to 0$). To simplify the procedure for integrating over a thin transition layer, we assume that the diffusion coefficients $D(x,t)$ far from the interface are constant:

$$D(x \leq s(t) - \varepsilon, t) = D^{\alpha} = const_1,$$
$$D(x \geq s(t) + \varepsilon, t) = D^{\beta} = const_2.$$

In this case, we represent the diffusion coefficient $D(x,t)$ in a model form – the sum of the Heaviside functions:

$$D(x,t) =$$
$$= D^{\alpha}\theta(s(t) - x + \varepsilon) + D^{\beta}\theta(x - s(t) + \varepsilon), \quad (9)$$

where $\theta(x)$ – asymmetric identity function that satisfies the conditions $\theta(x) = 1$ if $x \geq 0$, $\theta(x) = 0$ if $x < 0$.

The graph of the dependence of the diffusion coefficient of the medium $D(x - s(t), t)$ on the coordinate x for the boundary between the phases $x = s(t)$ is shown in Fig. 2.

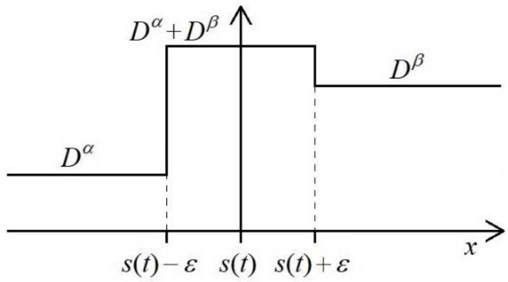

Fig. 2. Model dependence of the coefficient diffusion of the medium $D(x,t)$ on the coordinate x and phase boundaries $x = s(t)$

The summation of the diffusion coefficients of both phases in the transition layer (9) is associated with the assumption that the superimposition of the diffusion motion of particles does not lead to their extinction, but, on the contrary, intensifies it, i.e., leads to an increase in the diffusion coefficient. The assumption about the summation of characteristic parameters in the transition layer, for example, the viscosity of contacting liquid media, turned out to be productive in describing the Kelvin-Helmholtz (KH) instability [10]. The use of such an approach made it possible to lower the theoretical limit of the threshold rate of the onset of KH instability to the experimental one. However, for the surface tension coefficients of two contacting liquids, this approach is not applicable. The surface tension coefficients in the transition layer are subtracted according to the Antonov rule [11], which is confirmed experimentally.

The representation of the jumping diffusion coefficients of two phases as a step function allows us to write one diffusion equation of the form (1) with a coordinate-dependent total diffusion coefficient.

Let us integrate the function equal to zero

$$G(x,t) = \frac{\partial}{\partial t}N(x,t) - \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial}{\partial x}N(x,t)\right) = 0,$$

within infinite limits:

$$\int_{-\infty}^{+\infty}\left[\frac{\partial}{\partial t}N(x,t) - \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial}{\partial x}N(x,t)\right)\right]dx =$$
$$\int_{-\infty}^{+\infty}\left[\frac{\partial}{\partial t}N(x,t) - \left(\frac{\partial}{\partial x}D(x,t)\right)\left(\frac{\partial}{\partial x}N(x,t)\right) -\right.$$
$$\left.-D(x,t)\frac{\partial^2}{\partial x^2}N(x,t)\right]dx = 0. \qquad (10)$$

1. Calculate the sum of the first and third integrals of equality (10).

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial t}N(x,t)dx - \int_{-\infty}^{\infty}D(x,t)\ \frac{\partial^2}{\partial x^2}N(x,t)dx =$$
$$= \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial t}N(x,t) - D(x,t)\ \frac{\partial^2}{\partial x^2}N(x,t)\right]dx =$$
$$= \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{s(t)-\varepsilon}\left[\frac{\partial}{\partial t}N(x,t) - D(x,t)\ \frac{\partial^2}{\partial x^2}N(x,t)\right]dx +\right.$$
$$+\int_{s(t)-\varepsilon}^{s(t)+\varepsilon}\left[\frac{\partial}{\partial t}N(x,t) - D(x,t)\ \frac{\partial^2}{\partial x^2}N(x,t)\right]dx +$$
$$+\int_{s(t)+\varepsilon}^{\infty}\left[\frac{\partial}{\partial t}N(x,t) -\right.$$
$$\left.\left.- D(x,t)\ \frac{\partial^2}{\partial x^2}N(x,t)\right]dx\right)\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{s(t)-\varepsilon}\left[\frac{\partial}{\partial t}N^{\alpha}(x,t) -\right.\right.$$
$$\left.- D^{\alpha}(x,t)\ \frac{\partial^2}{\partial x^2}N^{\alpha}(x,t)\right]dx + \int_{s(t)-\varepsilon}^{s(t)+\varepsilon}\left[\frac{\partial}{\partial t}N(x,t) -\right.$$
$$\left.- D(x,t)\ \frac{\partial^2}{\partial x^2}N(x,t)\right]dx \quad +$$
$$\left.+\int_{s(t)+\varepsilon}^{\infty}\left[\frac{\partial}{\partial t}N^{\beta}(x,t) - D^{\beta}(x,t)\ \frac{\partial^2}{\partial x^2}N^{\beta}(x,t)\right]dx\right) =$$
$$= \lim_{\varepsilon\to 0}\left(\int_{s(t)-\varepsilon}^{s(t)+\varepsilon}\left[\frac{\partial}{\partial t}N(x,t) - D(x,t)\ \frac{\partial^2}{\partial x^2}N(x,t)\right]dx.\right.$$

Expand N(x,t) in a Taylor row in ambit x = s(t):

$$N(x,t) = N(s(t),t) + \frac{\partial}{\partial x}N(s(t),t)\big(x - s(t)\big) +$$
$$+\frac{1}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial x}N(s(t),t)\big(x - s(t)\big)^2 + \ldots$$

. In this case, the following inequality $x - s(t) = \delta \ll l$ is true.

$$\lim_{\varepsilon\to 0}\left(\int_{s(t)-\varepsilon}^{s(t)+\varepsilon}\left(\frac{\partial}{\partial t}N(s(t),t) +\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial t}\frac{\partial}{\partial x}N(s(t),t)\big(x - s(t)\big) + \cdots\right)dx\right) \approx$$
$$\approx \lim_{\varepsilon\to 0}\left(\int_{s(t)-\varepsilon}^{s(t)+\varepsilon}\left(\frac{\partial}{\partial t}N(s(t),t) +\right.\right.$$
$$\left.\left.+\frac{\partial}{\partial x}N(s(t),t)\frac{\partial s(t)}{\partial t}\right)dx\right).$$

In the second integral, we can replace $N(s(t),t)$ with *N(x, t)*, since it was noted earlier that $x \approx s(t)$. Then:

$$\lim_{\varepsilon\to 0}\left(\frac{\partial}{\partial t}N(s(t),t)\Big|_{s(t)-\varepsilon}^{s(t)+\varepsilon} +\right.$$
$$\left.+\frac{\partial s(t)}{\partial t}\int_{s(t)-\varepsilon}^{s(t)+\varepsilon}\frac{\partial}{\partial x}N(x,t)\,dx\right) =$$
$$= \lim_{\varepsilon\to 0}\left(\frac{\partial}{\partial t}N(s(t),t)\cdot 2\varepsilon + \frac{\partial s(t)}{\partial t}N(x,t)|_{s(t)-\varepsilon}^{s(t)+\varepsilon}\right) =$$
$$= \frac{\partial s(t)}{\partial t}\left(N^{\beta} - N^{\alpha}\right). \qquad (11)$$

2. Calculate the second integral of the equality (10):

$$\int_{-\infty}^{+\infty}\left(\frac{\partial}{\partial x}D(x,t)\right)\left(\frac{\partial}{\partial x}N(x,t)\right)dx =$$
$$\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\left(D^{\beta}\delta(x - s(t) + \varepsilon) - D^{\alpha}\delta(x -\right.$$
$$\left.s(t) - \varepsilon)\right)\frac{\partial}{\partial x}N(x,t)dx =$$
$$D^{\beta}\frac{\partial}{\partial x}N^{\alpha}(s(t),t) - D^{\alpha}\frac{\partial}{\partial x}N^{\beta}(s(t),t). \qquad (12)$$

Thus, expression (10), after integration, takes the form:

$$\int_{-\infty}^{+\infty}G(x,t)dx = \frac{\partial s(t)}{\partial t}\left(N^{\beta}(x,t) - N^{\alpha}(x,t)\right) -$$
$$-D^{\beta}\frac{\partial}{\partial x}N^{\alpha}(s(t),t) + D^{\alpha}\frac{\partial}{\partial x}N^{\beta}(s(t),t) = 0. \qquad (13)$$

From (13) it follows that the rate of movement of the interface in the modified representation is equal to:

$$\frac{\partial s(t)}{\partial t} = \frac{D^{\beta}\frac{\partial}{\partial x}N^{\alpha}(s(t),t) - D^{\alpha}\frac{\partial}{\partial x}N^{\beta}(s(t),t)}{N^{\beta} - N^{\alpha}}. \qquad (14)$$

The standard Stefan condition (see, for example, [1]) is that the motion velocity IB is proportional to the difference between the products of the diffusion coefficient of the β phase and the derivative of the concentration of the β phase with respect to the coordinate and the product of the diffusion coefficient of the α phase with the derivative of the concentration of the α phase with respect to the coordinate, t.e. is proportional to the quantity $D^{\beta}\frac{\partial}{\partial x}N^{\beta}(s(t),t) - -D^{\alpha}\frac{\partial}{\partial x}N^{\alpha}(s(t),t)$. The difference between the modified Stefan condition (14) and the standard one is that in the latter, the derivatives of the concentrations with respect to the coordinate $\alpha(\beta)$ of the phases are interchanged.

Expressions (7), (8), (14) imply the modified Stefan condition:

$$\frac{\partial s(t)}{\partial t} = -\frac{D^\beta N_2^\alpha \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{tD^\alpha}} e^{-\frac{(s(t))^2}{4tD^\alpha}}}{N^\beta - N^\alpha} + \\ + \frac{D^\alpha N_2^\beta \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{tD^\beta}} e^{-\frac{(s(t))^2}{4tD^\beta}}}{N^\beta - N^\alpha}. \qquad (15)$$

By substituting into (15) $s(t) = A\sqrt{t}$, we obtain a transcendental equation to determine A:

$$A = -\frac{2}{\sqrt{\pi}} \frac{D^\beta N_2^\alpha \frac{1}{\sqrt{D^\alpha}} e^{-\frac{A^2}{4D^\alpha}}}{N^\beta - N^\alpha} + \frac{2}{\sqrt{\pi}} \frac{D^\alpha N_2^\beta \frac{1}{\sqrt{D^\beta}} e^{-\frac{A^2}{4D^\beta}}}{N^\beta - N^\alpha}. \qquad (16)$$

Let us rewrite (16) in a form more convenient for application to experimental data by making the following substitutions $y = A/2\sqrt{D^\beta}$, $\lambda_{\alpha\beta} = D^\alpha/D^\beta$, we obtain the equation:

$$y = \\ = \frac{\lambda_{\alpha\beta}}{\sqrt{\pi}\sqrt{\lambda_{\alpha\beta}}} \frac{\frac{\left(1 - \frac{N_s}{N^\alpha}\right)}{\operatorname{erf}\left(\sqrt{\lambda_{\beta\alpha}} y\right)} \frac{\sqrt{\lambda_{\beta\alpha}}}{\lambda_{\alpha\beta}} e^{-\lambda_{\beta\alpha} y^2}}{1 - \frac{N^\beta}{N^\alpha}} + \\ + \frac{\lambda_{\alpha\beta}}{\sqrt{\pi}\sqrt{\lambda_{\alpha\beta}}} \frac{\frac{\frac{N_s}{N^\alpha} - \frac{N_\beta}{N^\alpha}}{\operatorname{erf}(y)} e^{-y^2}}{1 - \frac{N^\beta}{N^\alpha}}, \qquad (17)$$

where $\lambda_{\alpha\beta} = 1/\lambda_{\beta\alpha}$, $N^\alpha > N^\beta$.

We assume in equation (17) that the inequalities $A/\sqrt{D^\beta} \ll 1, \lambda_{\alpha\beta} \ll 1$, is true and find its solution for the case $y \ll 1$.

Then from equation (17) we find the value of the coefficient A:

$$A \approx \sqrt{2D^\beta \sqrt{\lambda_{\beta\alpha}} \frac{1 - \frac{N_s}{N^\alpha}}{1 - \frac{N^\beta}{N^\alpha}}}. \qquad (18)$$

DISCUSSION OF THE OBTAINED RESULTS

Let us check the applicability of the obtained value of the coefficient (18) on the experimental results on the displacement of the Cu/Sn interface during diffusion bonding as a result of isothermal annealing at temperatures of 433…473 K [12]. In this work, it is shown that, as a result of annealing, composite layers (compounds) consisting of Cu_3Sn and Cu_6Sn_5 are formed at the Cu/Sn interface. The boundaries between $Cu/Cu_3Sn/Cu_6Sn_5/Sn$ move in proportion to t^n, where the exponent n depending on the temperature of 433, 453, 473 K is 0.37, 0.43, 0.5, respectively.

To verify the agreement between the results of the theoretical model and the experiment, we select the data for annealing the samples at a temperature of 473 K in an oil bath with silicone oil. The choice of data at such an annealing temperature is due to the fact that in this case diffusion along grain boundaries is excluded, and there is only volume diffusion, which is considered in the proposed model. It follows from the experiments that the displacement of the interfacial boundary of the total thickness of two layers of the compound l increases monotonically with increasing annealing time t according to law $l = k(t/t_0)^{0.5}$, where $t_0 = 1\ s$ – unit of time, one second, k – a constant having the dimension of length in SI – m, one meter. In this case, the layers of compounds move with time according to one law.

To calculate the constant A, which in [12] is denoted k, let us set the values of the diffusion coefficients on both sides of the transition layer. Denote D^α diffusion coefficient at the boundary Cu/Cu_3Sn [13]: $D^\alpha = D^*_{\mathrm{Cu/Cu_3Sn}} = 3.53 \cdot 10^{-17}$ m²/s, and D^β – diffusion coefficient at the boundary Sn/Cu_6Sn_5: $D^\beta = D^*_{\mathrm{Sn/Cu_6Sn_5}} = 2.37 \cdot 10^{-16}$ m²/s. Concentration ratio N^β/N^α determined by the ratio of densities: $N^\beta/N^\alpha = 7.3/8.94 = 0.82$.

Substitution in (18) of the values of the diffusion coefficients D^α and D^β gives the value of the coefficient A:

$$A = 3.98 \cdot 10^{-8} \cdot \sqrt{\frac{\left(1 - \frac{N_s}{N^\alpha}\right)}{0.09}}. \qquad (19)$$

It follows from the experimental data [12] that for the total total thickness of the compound, the dimensionless constant k/k_0, where $k_0 = 1$ m – unit of length 1 m, determined by the value $k = 1.69 \cdot 10^{-8}$. It follows from (18) that for the ratio of order constants $N_s/N^\alpha \approx 0.985$ constant value A is close to experimental value k.

Thus, the comparison of the theoretical consideration of the SP using a new approach in describing the motion of the interfacial boundary with experimental data indicates the validity of the proposed method for obtaining the Stefan condition. This conclusion, under certain assumptions, is based on the correspondence between the theoretically calculated distance of displacement of the interface and the experimental one.

Substitution of the experimental data [12, 13] into the standard (not modified) Stefan condition gives a negative value of the coefficient A:

$$A = \frac{2}{\sqrt{\pi}} \left(\frac{D^\beta \frac{(N_s - N_\beta)}{N^\alpha \operatorname{erf}(y)} \frac{1}{\sqrt{D^\beta}} e^{-\frac{A^2}{4D^\beta}}}{\frac{N^\beta}{N^\alpha} - 1} + \\ + \frac{D^\alpha \frac{(N_\alpha - N_s)}{N^\alpha \operatorname{erf}\left(\sqrt{\lambda_{\beta\alpha}} y\right)} \frac{1}{\sqrt{D^\alpha}} e^{-\frac{A^2}{4D^\alpha}}}{\frac{N^\beta}{N^\alpha} - 1} \right) < 0.$$

Hence it follows that IB moves in a direction that is opposite to that observed in the experiment.

This result speaks in favor of using the modified Stefan condition, which describes the motion of IB in problems of heat conduction or diffusion.

CONCLUSIONS

The article analyzes the state of the art in research related to diffusive phase transformation in solids, both as a result of heat transfer and diffusion of particles. Such problems are modeled within the framework of the two-phase SP with the boundary between the phases moving with time. The position of the moving boundary between the phases is determined by the Stefan condition, which follows from the condition of equality of heat or particle fluxes on both sides of the phases. This condition can be obtained from the original diffusion equation, in which the diffusion coefficient should be considered to be coordinate-dependent, by integrating over a thin transition layer, and by tending the layer thickness to zero. This layer includes an interfacial boundary, where the value of the diffusion coefficient changes abruptly from one value to another. If we assume that the diffusion coefficient in the transition layer is equal to zero, then we arrive at the standard and thoroughly studied SP with a moving interface.

In the present work, the assumption is made that the diffusion coefficients of both phases in the transition layer are summed, and the modified Stefan condition is obtained by the integration method. It is shown that, as in the standard SP, the displacement of the interface is proportional to the square root of time.

It is shown that the modified Stefan condition differs from the standard one in that in the latter, the derivatives of the concentrations with respect to the coordinate α(β) of the phases are interchanged.

To verify the validity of the modified Stefan condition obtained in this work, we used the experimental results on the displacement of the Cu/Sn interface during diffusion bonding as a result of isothermal annealing. Under certain assumptions about the parameters of the Cu/Sn interface, a good quantitative agreement was obtained between the results arising from the modified Stefan condition and the experimental results.

Comparison of the interface displacement, which follows from the standard SP, with the experimental data leads to a contradiction: the theoretically calculated displacement of the interface is directed in the direction opposite to that observed in the experiment.

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МОДИФІКОВАНА УМОВА СТЕФАНА В ЗАДАЧІ СТЕФАНА

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Розглянуто двофазну одновимірну задачу Стефана (ЗС) з межею між фазами, що переміщується з часом. Положення межі визначається модифікованою умовою Стефана (МУС), яка отримана з вихідного нелінійного рівняння дифузії методом інтегрування по тонкому перехідному шару і устремлінням його товщини до нуля. При отриманні МУС коефіцієнт дифузії представлений сумою ступінчастих функцій Хевісайду. Показано, що МУС відрізняється від стандартної тим, що у неї похідні концентрації фаз по координаті змінюються місцями. Отримано вираз для переміщення міжфазного кордону, яке, як і у стандартній ЗС, пропорційне квадратному кореню з часу. Результати використання МУС підтверджуються експериментальними даними щодо переміщення межі Cu/Sn при дифузійному з'єднанні при ізотермічному відпалі.