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LAW OF LARGE NUMBERS OF PROBABILITY THEORY AND FORMATION OF THE ENERGY SPECTRUM OF SYNCHROTRON RADIATION OF RELATIVISTIC ELECTRON IN THE STORAGE RING

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The fluctuations of synchrotron radiation (SR) energy losses during the motion of relativistic electrons in the uniform magnetic field in a storage ring are considered. The influence on the formation of the energy spectrum of the SR photon flux of the duration of registration of SR relativistic electrons is analyzed. It has been established that with the increase in the duration of temporal registration, the energy spectrum of the SR photon flux is regularly transformed into the asymptotically normal density in accordance with the law of large numbers of probability theory. It is determined that the main mechanism in the formation of the energy spectrum is successive convolutions of the current spectra.

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INTRODUCTION

The cyclic motion of relativistic electrons is accompanied by the emission of a stream of synchrotron radiation (SR) photons [1, 2]. Spontaneous emission is the Poisson process, therefore, the monoenergetic electron, as it revolves and emits, will have an energy spectrum, and the dispersion of this spectrum will increase with time. The energy spectrum of electron $p(E_e)$ will correspond to the distribution density of the energy $F(E_\gamma)$, stored in the SR field. In this paper, the statistical mechanism for the formation of the energy density of the distribution of the SR quanta flux is considered and its evolution in the process of emission into the normal law is traced

MATHEMATICAL MODEL

The space surrounding the emitting electron is a detector that absorbs the energy stored in the SR field. The probability that energy E_k will be emitted in the range $(\varepsilon_k, \varepsilon_k + \Delta\varepsilon_k)$ of photon energies is [3, 4]

$$P(E_k = m\varepsilon_k) = \frac{\left[\Delta\varepsilon n(\varepsilon_k)\right]^m}{m!}\exp\left[-\Delta\varepsilon n(\varepsilon_k)\right], \quad (1)$$

where $n(\varepsilon)$ is the spectral density of the SR photon flux, that is, it is described in terms of the Poisson distribution. Consider the probability of radiation on the SR of the certain energy E_γ, that is $E_\gamma = \sum_k E_k$. This quantity is random $F(E_\gamma)$ and is characterized by the probability distribution density $F(E_\gamma)$, or, equivalently, the generating function $Q_{E_\gamma}(\lambda)$, which is the Fourier transform of the density $F(E_\gamma)$.

Since energy registration is a time-additive process, the transformant $Q_{E_k}(\lambda)$ is structurally a product of:

$$Q_{E_\gamma}(\lambda) = \prod_k Q_{E_k}(\lambda) =$$

$$= \exp\left\{\sum_k \ln\left[\sum_{m=0}^{\infty} P(E_k = m\varepsilon_k)\exp\left[-i\lambda m\varepsilon_k\right]\right]\right\}. \quad (2)$$

The SR spectrum of the relativistic electron consists of a large number of harmonics, so in (2) one can go from summation over k to integration over ε. Then, applying the Fourier transform, we write:

$$F(E_\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \exp\left(i\lambda E_\gamma\right)\times$$

$$\times\exp\left\{-\int_0^{\infty} d\varepsilon n(\varepsilon)\left[1-\exp(-i\lambda\varepsilon)\right]\right\}. \quad (3)$$

The result similar to (3) was obtained earlier in [5]. This is due to the statistical equivalence of the phenomena under consideration, since the electron losses in the SR in the storage ring are compensated by its high-frequency system.

For further consideration, it turns out to be convenient to switch to relative (normalized) variables. For this purpose, we use the representation for the spectral density of the SR photon flux [1, 3], which takes place if the energy of the emitted photons is small compared to the electron energy E_e:

$$n(\varepsilon) = \frac{9\sqrt{3}}{8\pi}\frac{\alpha}{\varepsilon_C}\int_{\varepsilon/\varepsilon_C}^{\infty} d\eta K_{5/3}(\eta). \quad (4)$$

In density (4) $K_{5/3}(\eta)$ is the Macdonald function of the index 5/3, ε_C is the energy of the characteristic (equivalent) quantum, $\varepsilon_C = 1{,}5\gamma^3\hbar\omega_L$, γ is the relativistic factor, ω_L is the Larmor frequency. Also in (4) the parameter α is introduced. The physical meaning of this parameter is that it determines the number of equivalent quanta ε_C with energy contained in the array-averaged realizations over the duration of registration of the energy emitted by the electron, i.e.

$$\left\langle E_\gamma\right\rangle = \int_0^{\infty} dE_\gamma F(E_\gamma)\, E_\gamma = \alpha\varepsilon_C. \quad (5)$$

The expressions for the density $F(E_\gamma)$ and the generating function $Q_{E_\gamma}(\lambda)$ will take on a universal form (not explicitly dependent on the energy of the relativistic electron) if we pass to the variables $w = E_\gamma / \left\langle E_\gamma\right\rangle$ and $q = \lambda\left\langle E_\gamma\right\rangle$:

$$F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \exp(iqw) Q_w(q), \quad (6)$$

wherein

$$Q_w(q) = -\frac{9\sqrt{3}}{8\pi} i \int_0^\infty dz \int_0^\infty dy \int_y^\infty dx\, K_{5/3}(x) \exp\left(-i\frac{yz}{\alpha}\right). \quad (7)$$

After taking two of the three integrals in (7), we can obtain the following integral representation for the desired density

$$F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \exp(iqw)\, Q_w(q), \quad (8)$$

where

$$Q_w(q) = \exp\left\{\frac{9\alpha}{4\sinh(2q/\alpha)}\left[\frac{1}{\sqrt{12}}A(q)+iB(q)\right]\right\},$$

$$A(q) = 5\sinh\left(2\operatorname{asinh}\left(\frac{q}{\alpha}\right)\right) - 6\sinh\left(\frac{5}{3}\operatorname{asinh}\left(\frac{q}{\alpha}\right)\right),$$

$$B(q) = \cosh\left(\frac{5}{3}\operatorname{asinh}\left(\frac{q}{\alpha}\right)\right) - \cosh\left(\operatorname{asinh}\left(\frac{q}{\alpha}\right)\right).$$

Integral representation (8) gives the desired analytical expression for the density $F(E_\gamma)$ of the SR energy spectrum of relativistic electron.

SR ENERGY SPECTRUM RELATIVISTIC ELECTRON IN STORAGE RING

On Fig. 1 shows the result of calculating the density $F(w)$ for different values α =1, α =2, and α =3. Dependencies are given for the relative argument $w = E_\gamma / \langle E_\gamma \rangle$. It can be seen that for sufficiently small α the density $F(w)$ corresponds to the spectral density of the photon flux SR.

One can see how spectrum $F(w)$ is formed with an increase in the average absorbed energy (number of equivalent photons with energy ε_C) according to (5). Due to the broadband nature of the original spectrum $n(\varepsilon)$, the emerging density $F(w)$ basically repeats such properties $n(\varepsilon)$ as single-top and zero at the periphery.

On Fig. 2 shows the similar result of density $F(w)$ calculation but for values α =5, α =10, and α =15. It can be seen how, as the result of multiple convolution operations, the density $F(w)$ acquires the properties of normal density.

The further transition to normal density is illustrated in Fig. 3, the dependences of which are calculated at values α =50, α =100, and α =150.

From Figs. 1-3 it can be seen that if the number of absorbed equivalent energy quanta is large ε_C, that is, $\alpha \gg 1$, then $F(w)$ has the Gaussian form. In the limit $\alpha \to \infty$, from (8) one can obtain the Gaussian asymptotic with dispersion $\sigma_w^2 \sim \alpha^{-1}$ which corresponds to one of the representations of the delta-function $\delta(w-1)$. For the small number of equivalent photons in the average absorbed energy $\langle E_\gamma \rangle$, the distribution density $F(w)$ asymptotically, as it decreases α, focuses around the vertical and horizontal axes (see Fig. 1), passing at $\alpha \to 0$ to $n(\varepsilon)$.

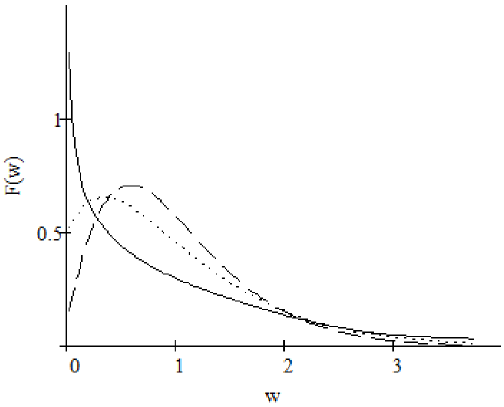

Fig. 1. Distribution densities $F(w)$ absorbed SR energy by the ideal detector for a different number $\alpha = \langle E_\gamma \rangle \varepsilon_C^{-1}$ equivalent quanta with energy ε_C; α =1 (line), α =2 (dots), α =3 (dotted line)

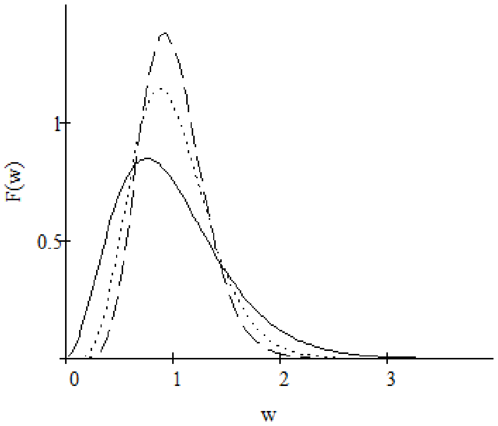

Fig. 2. Distribution densities $F(w)$ absorbed SR energy by an ideal detector; α =5 (line), α =10 (dots), α =15 (dotted line)

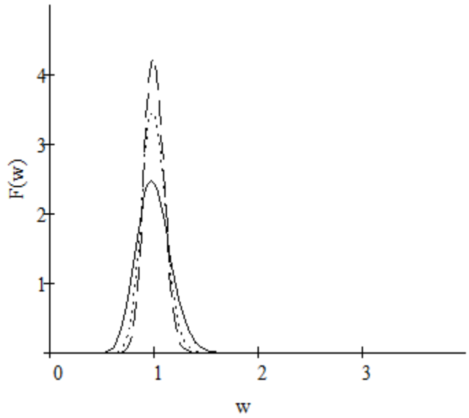

Fig. 3. Distribution densities $F(w)$ absorbed SR energy by an ideal detector; α =50 (line), α =100 (dots), α =150 (dotted line)

From expression (8) for $F(w)$ it follows that variance $\sigma_w^2 = \langle w^2 \rangle - \langle w \rangle^2 = \langle w^2 \rangle - 1$ is equal to

$$\sigma_w^2 = -\frac{d^2}{dq^2}\beta(q)\bigg|_{q=0} = \frac{55\sqrt{3}}{72}\alpha^{-1}, \quad (9)$$

therefore, for the dispersion of the absorbed SR energy, we obtain

$$\sigma_\gamma^2 = \langle E_\gamma^2 \rangle - \langle E_\gamma \rangle^2 = \langle E_\gamma \rangle^2 \sigma_w^2 = \frac{55\sqrt{3}}{72}\alpha\varepsilon_C^2. \quad (10)$$

The number of photons absorbed by the detector α is determined by the equivalent quanta α_{eq} emitted by the electron in one revolution and the registration conditions

$$\alpha = \alpha_{\mathrm{eq}} fT\Delta\psi , \qquad (11)$$

where $\alpha_{\mathrm{eq}} = 4e_0^2\gamma / 9c\hbar$ (e_0 is the charge of an electron, c is the speed of light, f is the frequency of revolution, T is the duration of one exposure, during which the total absorbed energy E_γ is recorded, $\Delta\psi$ is the factor due to the aperture of the receiving device).

From (8) and Fig. 1 it follows that a non-trivial (non-Gaussian) form of the function $F(w)$ is realized for small α, $\alpha \le 50$. For an electron with an energy of 100 MeV, the number of emitted equivalent photons per revolution is $\alpha_{\mathrm{eq}} = 0.5$. Choosing $\Delta\psi = 10^{-4}$ rad, we get that if one electron circulates in a magnetic field [6], and the duration of registration is several hundred thousand revolutions, then the number of registered equivalent photons is equal to $\alpha_{\mathrm{eq}} = 25$.

RADIATING RELATIVISTIC ELECTRON

For the emitting electron, the detector is the surrounding space. The energy spectrum $p(E_e)$ of an electron beam emitting during circulation in a magnetic field can be obtained from the relationship [7, 8]

$$p(E_e) = \int dE_e' \, p'(E_e')F(E_e' - E_e) , \qquad (12)$$

where $p'(E_e')$ is the energy spectrum of the electron beam at the initial moment of time. For example, if

$$p'(E_e') = (2\pi s^2)^{-1/2} \exp\left[-\left(E_e' - \langle E_e' \rangle\right)^2 / 2s^2\right] \qquad (13)$$

s a normal law with dispersion s^2, then we obtain, taking into account that $E_e \gg E_\gamma$

$$p'(E_e') = \frac{1}{\left[2\pi\left(s^2 + \langle E_\gamma \rangle^2\right)\right]^{1/2}} \times$$
$$\times \exp\left[-\frac{\left(E_e' - \langle E_e' \rangle + \langle E_\gamma \rangle\right)^2}{2\left(s^2 + \langle E_\gamma \rangle^2\right)}\right], \qquad (14)$$

where $\langle E_\gamma^2 \rangle = \langle E_\gamma \rangle^2 \left(1 + \sigma_w^2\right)$. From (14) it follows that

$$\langle E_e \rangle = \langle E_e' \rangle - \langle E_\gamma \rangle , \qquad (15)$$

and in the adiabatic approximation for the increase in the dispersion of the energy spectrum we have

$$\Delta s^2 = \varepsilon_C^2 \alpha \, f\Delta T \left(\alpha f \Delta T + 55\sqrt{3}/72\right), \qquad (16)$$

that is, the magnitude of the energy spread of the radiating electron grows rapidly with an increase in its energy.

CONCLUSIONS

Thus, the effect of the discreteness of spontaneous emission of SR quanta on the energy distribution density of the synchrotron radiation field has been considered, and the time evolution of the energy spectrum of relativistic electron as it moves in the uniform magnetic field has been studied. The limits of applicability of the obtained results are limited by the approximation that meets the condition that the energy of the emitted equivalent SR quantum is small compared to the electron energy [2, 3], which is realized in modern cyclic accelerators. It follows from the study that with an increase in the duration of temporal registration, the energy spectrum of the SR photon flux is regularly transformed into an asymptotically normal density in accordance with the law of large numbers of probability theory. It is determined that the main mechanism in the formation of the energy spectrum is successive convolutions of the current spectra.

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ЗАКОН ВЕЛИКИХ ЧИСЕЛ ТЕОРІЇ ЙМОВІРНОСТЕЙ І ФОРМУВАННЯ ЕНЕРГЕТИЧНОГО СПЕКТРА СИНХРОТРОННОГО ВИПРОМІНЮВАННЯ РЕЛЯТИВІСТСЬКОГО ЕЛЕКТРОНУ В НАКОПИЧУВАЧІ

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Розглянуто флуктуації енергетичних втрат на синхротронне випромінювання (СВ) під час руху релятивістських електронів у однорідному магнітному полі у накопичувачі. Проаналізовано вплив на формування енергетичного спектра потоку квантів СВ тривалості реєстрації СВ релятивістських електронів. Встановлено, що зі збільшенням тривалості часової реєстрації енергетичний спектр потоку квантів СВ регулярно перетворюється на асимптотично нормальну щільність відповідно до закону великих чисел теорії ймовірностей. Визначено, що основним механізмом у формуванні енергетичного спектра є послідовні згортки поточних спектрів.