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CHAOTIC REGIMES GENERATED BY SPECIAL POINTS AND SINGULAR SOLUTIONS

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The role of singular solutions and some singular points of ordinary differential equations with respect to the occurrence of chaotic regimes in physical systems is considered. It is shown that in those physical systems in which singular solutions exist, they can cause the occurrence of chaotic regimes. Examples of such systems are given. Of particular interest are the results showing that chaotic regimes can occur even in systems with one degree of freedom and even in completely integrable systems. These results, at first glance, violate known mathematical theorems (Poincare-Bendixson), and also destroy the paradigms of dynamic chaos. It is shown that this contradiction is an apparent contradiction. The reason for the appearance of chaotic dynamics in these cases are special solutions. At the points of this solution, the conditions of the uniqueness theorem are violated. In numerical calculations, this is manifested in the fact that a random component (numerical noise) appears in the calculations. As a result, the system under study becomes a system with 1.5 degrees of freedom. It is shown that in almost all cases, increasing the accuracy of the calculations allows suppressing chaotic dynamics in systems with one degree of freedom.

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INTRODUCTION

First of all, it is necessary to say a few words about special solutions that can generate chaotic regimes. Some confusion may arise here. Indeed, in English, special solutions (singular solutions) are solutions that arise in various areas of scientific research. First of all, this concerns mathematics. There are a lot of mathematical works devoted to the study of various special solutions. A significantly smaller number of works are devoted to physical research, in which special solutions are often understood as solutions that are themselves singular (they tend to infinity). Many studies are devoted to those cases when such special (singular) parameters appear in mathematical models of the physical system being studied. There is another area of research, the results of which are the least numerous, in which a special solution appears, at the points of which the conditions of the uniqueness theorem are violated.

All the solutions listed above are called singular solutions in English-language literature. In this paper, we will only touch upon the last case, i.e. the case, when studying physical systems, solutions of the last type arise (at the points of which the conditions for the uniqueness theorem are violated). As for singular points, in this paper we will only touch upon structurally unstable points. More specifically, those in which the decisive role is played by singular points of the saddle type.

It should also be noted that when analyzing mathematical models that describe physical processes, conditions are explicitly or implicitly imposed that exclude the consideration of such solutions, i.e. a condition of uniqueness of solutions is imposed. As an example, we can cite the definition of the main property of the phase space, in which the regular and stochastic dynamics of physical systems are considered in the book [1]:

"At any given moment of time, the trajectories in the phase space do not intersect...". This proposal immediately excludes the consideration of special solutions. In addition, this proposal immediately imposes the need for the implementation of a regime with dynamic chaos to have a system whose number of degrees of freedom is greater than or equal to 1.5. Indeed, if we have a system with one degree of freedom, then the phase space of such a system is a plane. If the phase trajectories cannot intersect, then the conditions for mixing these trajectories cannot be realized on the plane. This is also indicated by the Poincaré-Bendixson theorem.

One of the possible reasons why singular solutions do not receive due attention in physical research is the attitude of Steklov himself towards these solutions [2]. Indeed, even though such solutions and the algorithm for finding such solutions were introduced by Steklov himself, he expressed his attitude towards them in his works in the following way: "It goes without saying that singular solutions of various types are not always possible for any given system of differential equations. As practice shows, singular solutions are obtained only in exceptional cases. For example, all currently resolved questions of general mechanics and astronomy are obtained by integrating equations that do not admit any singular solutions. It is likely that all differential equations of mechanics do not have any singular solutions, although this proposition, which has important philosophical significance, as far as I know, remains unproven in the general case at present."

Below the material of the work is presented in three sections. The first one considers the problem, the results of the studies of which are presented in the works [3–5]. These works consider the physical problem of the dynamics of the Cartesian components of the magnetic moments of two interacting neutron stars. The authors managed to reduce the mathematical model of this interaction to the analysis of a mathematical model that has only one degree of freedom. The authors showed that chaotic regimes can exist in such a system. This result contradicts the basic paradigm of dynamic chaos, and contradicts the Poincare-Bendixson theorem. In this section, it is shown that the result obtained in these works is due to the presence of natural numerical noise in numerical studies.

In this case, the system becomes with 1.5 degrees of freedom. With a significant increase in the calculation accuracy, this noise decreases, and the chaotic regime disappears. The second section also considers a system with one degree of freedom, but which has a clearly expressed special solution. Chaotic regimes are also observed in this model. They disappear with high calculation accuracy. However, for their disappearance, a higher degree of calculation accuracy is needed than in the previous case. The third section presents the results of the analysis of particle dynamics at cyclotron resonances. This system, even in the simplest case, has two degrees of freedom (four ordinary differential equations). An important characteristic feature of the particle dynamics is discovered. This feature is that in the transverse plane of particle motion there is an extremely small region (a practically isolated point) through which all particle trajectories pass. This feature is characteristic of special solutions. Chaotic regimes of this motion are studied. The conclusion formulates the main results.

1. DIXON'S PROBLEM

A possible example of a physical problem in which special solutions appear is the system of equations introduced by Cummings et all [3] as a model of the dynamic behavior of the magnetic moments of a neutron star:

$$\begin{aligned}
&\frac{d}{d\tau}m_x = m_y - \frac{\lambda m_x m_z}{m_x^2+m_y^2+m_z^2} - \varepsilon m_x;\\
&\frac{d}{d\tau}m_y = -m_x - \frac{\lambda m_y m_z}{m_x^2+m_y^2+m_z^2} - \varepsilon m_y; \qquad (1)\\
&\frac{d}{d\tau}m_z = \frac{\lambda m_x m_z}{m_x^2+m_y^2+m_z^2} - \bar{\varepsilon} m_x - \left(\lambda - \bar{\varepsilon}\right),
\end{aligned}$$

where m_x, m_y, and m_z are the Cartesian components of the magnetization of the inner part of the neutron star sphere; τ – is the dimensionless time; λ is the dimensionless Landau damping parameter, and ε and $\bar{\varepsilon}$ are the dimensionless parameters of mechanical damping, perpendicular and parallel to the direction of the conserved angular momentum.

The conditions for the emergence of a chaotic regime will be the fulfilment of the following inequalities: ε/λ, $\bar{\varepsilon}/\lambda<1$.

Let's use the following transformations:

$$m_x \to x\cos\phi\,,\quad m_y \to x\sin\phi\,,\quad m_z \to z\ .$$

Then the number of related variables can be reduced to two:

$$\begin{aligned}
&\frac{d}{dt}x = \frac{xz}{x^2+z^2} - \alpha x;\\
&\frac{d}{dt}z = \frac{z^2}{x^2+z^2} - \beta z - (1-\beta); \qquad (2)\\
&\frac{d}{dt}\phi = 1,
\end{aligned}$$

where $\alpha = \varepsilon/\lambda$ $\beta = \bar{\varepsilon}/\lambda$.

Thus, the analysis of the dynamics of the system under consideration is reduced to the analysis of a system with one degree of freedom. According to the Poincaré-Bendixson theorem, such a system cannot have chaotic dynamics. This is also reported in the text of the paradigms of dynamic chaos. The authors claimed that such dynamics are due to the nature of the singular point, which they classified as a saddle-node. Below we will show that such a regime is due to the noise of the numerical calculation.

Analysing the equations (2) numerically, we can note the following features of the behaviour of this system. When chaotic motion occurs in the system, numerical calculation becomes possible only using methods with a constant step (Fig. 1). When using calculations with an adaptive step, the system cannot overcome a special point (Fig. 2). However, reducing the accuracy of calculations to (TOL=10^{-1}) leads to the fact that the calculation becomes possible (Fig. 3).

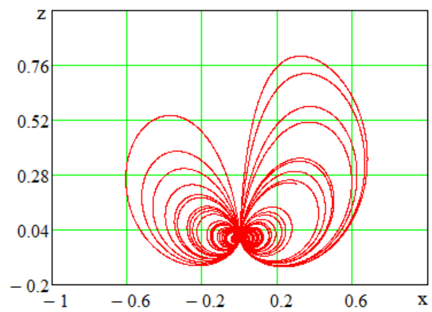

Fig. 1. Phase portrait corresponding to system (2). Constant step. Point (0.0) is singularity point ($x = 0.1, z = 0.1,\ \alpha = 0.6,\ \beta = 0.7$)

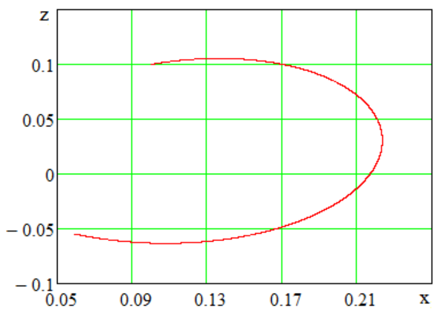

Fig. 2. Phase portrait corresponding to system (2). Adaptive step. Point (0.0) is singularity point. TOL=10^{-3} ($x = 0.1, z = 0.1,\ \alpha = 0.6,\ \beta = 0.7$).

The trajectory cannot overcome the singularity region

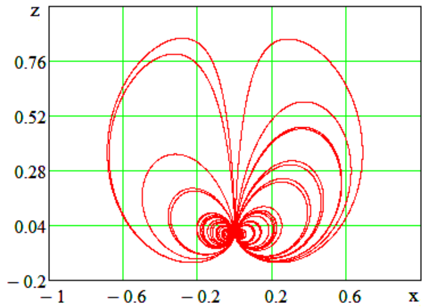

Fig. 3. Phase portrait corresponding to system (2). Adaptive step. Point (0.0) is singularity point. TOL=10[-1] ($x=0.1$, $z=0.1$, $\alpha=0.6$, $\beta=0.7$)

This behaviour of the system is easily explained by the fact that in the vicinity of the singular (special) point the density of trajectories increases. When we approach the singularity region in numerical calculation, the program (in the usual, standard mode) stops distinguishing these trajectories. With an increase in the calculation accuracy, the program moves to the singular point for a very long time. The higher the degree of accuracy of the program, the longer the program passes the singularity region. At the same time, the phase portraits of the system calculation by the method with a constant step (see Fig. 1) and the calculation by the method with an adaptive step, but with low accuracy (see Fig. 3) are fundamentally the same

2. OSCILLATOR WITH NONLINEAR FRICTION

Above, we considered an example in which the "chaotic" mode was caused by a singular point (saddle-node) and numerical computation noise. Strictly speaking, a saddle point cannot be a singular solution. It is isolated and the time to reach it is infinite. Therefore, in this section, we will consider a system with one degree of freedom, which has a singular point, which is simultaneously a singular solution of the system under consideration. This example is the model of an oscillator with nonlinear friction:

$$\dot{x}_0 = x_1, \qquad \dot{x}_1 = \left(\frac{x_1^2}{2x_0}\right) - 0.5x_0. \tag{3}$$

The dynamics of such a system was initially considered in [6]. This system has an integral

$$\varphi = \left(x_0 - R\right)^2 + x_1^2 - R^3 = 0, \tag{4}$$

where R – is the radius of the circles. Moreover, it can take arbitrary values. However, the centres of the circles are located on the axis $x_1 = 0$, and the radii of these circles are equal to the distance of these centres to the zero point. This point is common to all circles and is also a special solution of system (3). To prove this, we write out the general solution of system (3):

$$x_0 = R\left[1 \pm \sin(t + C)\right]. \tag{5}$$

In this case, using Steklov's algorithm [2], the value of the constant C can be set equal to

$$C = -t + \pi(n + 1/2), \; C = -t + \pi(n + 1/2). \tag{6}$$

Substituting this expression into the general solution (5), we find special solutions

$$x_{0s} = R\left[1 \pm 1\right] = \begin{cases} 0 \\ 2R \end{cases}. \tag{7}$$

Only one of them, namely $x_{0s} = 0$, satisfies the initial equations (3). Despite the apparent simplicity of the system (3), its integrability, the dynamics of the system can be irregular. Numerical calculations confirm this. Fig. 4 shows the phase portrait of the system in the form of circles. Fig. 5 shows the dependence of the variable x_0 on time. It is evident that the trajectory jumps from one circle to another randomly after passing the point of a special solution. Moreover, numerical calculations are very sensitive to the accuracy of the calculation and any change leads to a change in the time dynamics of the system. In addition, the correlation function quickly decreases (Fig. 6), which confirms the chaotic nature of the dynamics.

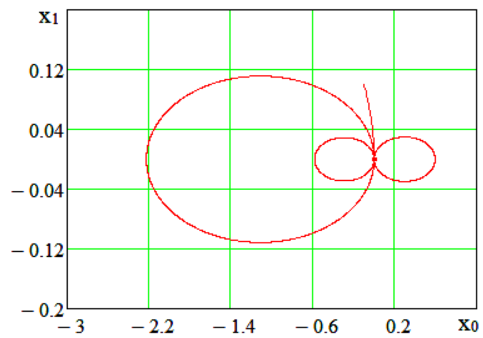

Fig. 4. Phase portrait of the system (3)

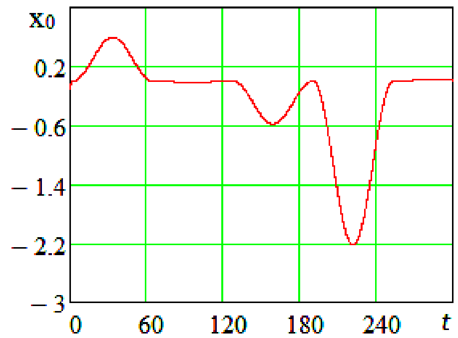

Fig. 5. Time dependence of the variable x_0. Jumps of trajectories from one circle to another are visible

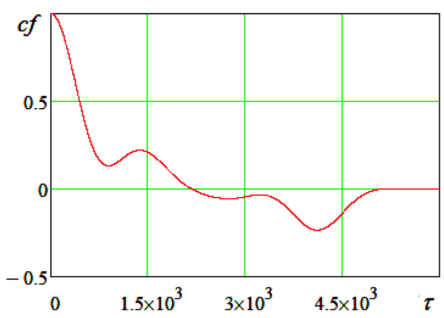

Fig. 6. Correlation function of the variable x_0

3. DYNAMICS AT CYCLOTRON RESONANCES

In this section, we consider a more complex system that describes the dynamics of charged particles moving in a wave field and in an external constant magnetic field. Such dynamics were described in many features in [7]. These dynamics are characterized by the fact that under certain conditions a chaotic regime can arise. In the plane of transverse particle motion, there is a small local region (practically a point) through which all particle trajectories pass. The density of trajectories in this region is very high. Therefore, when passing this region, particles can easily jump from one trajectory to another. The reason for such jumps may be insufficient accuracy of the numerical calculation, or the influence of self-consistent forces. Note that in this case, a system with two degrees of freedom is already considered (in the simplest case). Considering that special solutions represent the envelope of trajectories, we can say that in this case, too, special solutions play a decisive role in the emergence of chaotic particle dynamics. The example considered in the article [7] is complex. Below, we will present the results of considering the dynamics of a simpler system. In this system, all the features of particle dynamics that interest us are preserved.

Let us consider the motion of a charged particle in a constant magnetic field $\mathbf{H_0} = \{0, 0, H_{0z}\}$ and in the field of a plane electromagnetic wave with arbitrary polarization. The components of the electric and magnetic fields of such a wave can be represented as

$$\mathbf{E} = \mathrm{Re}(\mathbf{E}_0 e^{i\psi}), \ \mathbf{H} = \mathrm{Re}\left(\frac{1}{k_0}[\mathbf{kE}]\right), \qquad (8)$$

where $\psi \equiv \omega t - \mathrm{kr}$, $\mathbf{E}_0 = \alpha E_0$; $\alpha = \{\alpha_x, i\alpha_y, \alpha_z\}$ – is wave polarization vector; $k_0 = \omega / c$; ω, k – frequency and wave vector of the wave. Let us choose the simplest structure of the fields: $\vec{E} = \{0, E_y, 0\}$, $\vec{H} = \{0, 0, H_z\}$, $\vec{k} = \{k_x, 0, 0\}$. Let us introduce the following dimensionless variables:

$$\mathbf{p}_1 = \mathbf{p}/mc, \quad \mathbf{k}_1 = \mathbf{k}/k_0, \ \tau = \omega t,$$

$$\mathbf{r}_1 = k_0\mathbf{r}, \ H_1 = eH/mc\omega, \ \mathcal{E} = e\mathbf{E}_0/mc\omega, \ \mathbf{v}_1 = \mathbf{v}/c.$$

The equations of motion of particles will take the form:

$$\frac{d\vec{r}}{dt} = \frac{\vec{P}}{\gamma},$$

$$\frac{dP_x}{d\tau} = -k\frac{P_y}{\gamma}\left(-E_y \sin(\psi) + H_0\right), \qquad (9)$$

$$\frac{dP_y}{d\tau} = -E_y \sin(\psi) + k\frac{P_x}{\gamma}\left(-E_y \sin(\psi) + H_0\right).$$

The system of equations (9) was studied numerically. Fig. 7 shows the phase portrait of the system in the form of circles. It is important that all trajectories (circles) pass through a small neighbourhood of the zero point. Almost all trajectories have one common point – the envelope. This is the main characteristic of a special solution. Figs. 8 and 9 show the dependence of the variable x and y on time. It is evident that the trajectory jumps after passing the point of the special solution. In addition, the correlation function (Fig. 10) quickly decreases.

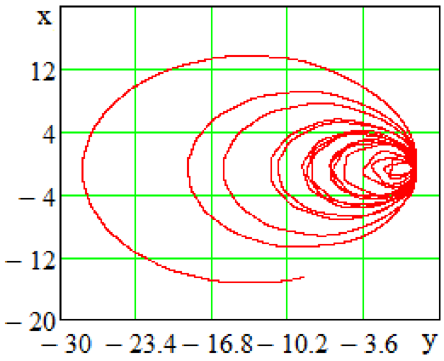

Fig. 7. Phase portrait of the system (9)

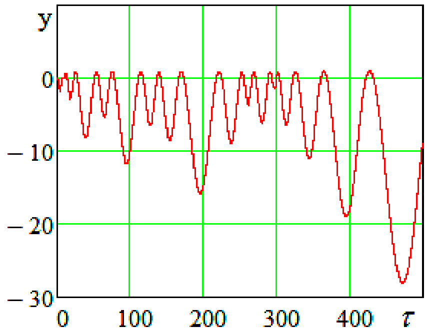

Fig. 8. Time dependence of the variable y

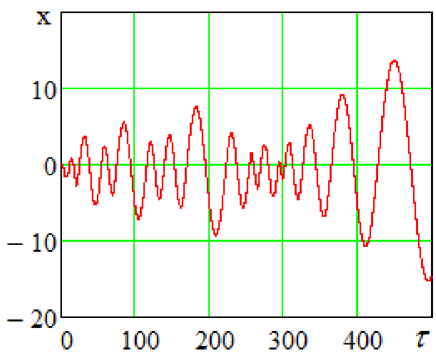

Fig. 9. Time dependence of the variable x

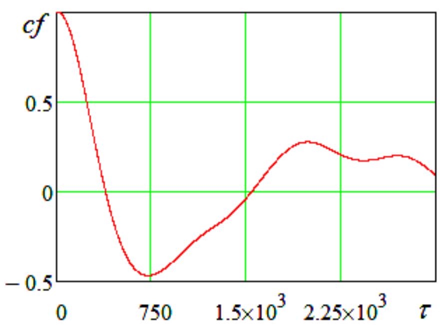

Fig. 10. Correlation function of the variable x

It is evident that the behaviour of the system is similar to the behaviour of systems in which the influence of special solutions on the dynamics of the system can be determined. This may indicate that in this system, too, the determining factor influencing the emergence of chaotic dynamics is special solutions.

However, in this case, increasing the degree of accuracy of numerical calculations does not change the main characteristics of the particle dynamics.

CONCLUSIONS

The most important conclusion that follows from the results described above is that it is necessary to find special solutions in the systems under study. If such exist, then the numerical calculation of such systems requires special attention. Indeed, as we have seen above, even in systems with one degree of freedom, chaotic regimes can manifest themselves. The presence of such regimes can significantly change the dynamics of the system under study. It should be noted that there are a significant number of works in which similar problems were studied using the point singular mapping method (see [8–10]).

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ХАОТИЧНІ РЕЖИМИ, ЯКІ ПОРОДЖЕНІ ОСОБЛИВИМИ ТОЧКАМИ І ОСОБЛИВИМИ РІШЕННЯМИ

В.О. Буц, В.В. Кузьмін

Розглядається роль особливих рішень і деяких особливих точок звичайних диференціальних рівнянь на виникнення хаотичних режимів у фізичних системах. Показано, що у тих фізичних системах, у яких існують особливі рішення, вони можуть спричинити хаотичні режими. Наведено приклади таких систем. Особливий інтерес представляють результати, в яких показано, що хаотичні режими можуть виникати навіть у системах з одним ступенем свободи і навіть у системах, що повністю інтегруються. Ці результати, здавалося б, порушують відомі математичні теореми (наприклад Пуанкаре-Бендиксона) і навіть руйнують парадигми динамічного хаосу. Показано, що ця суперечність є суперечністю, що здається. Причиною появи хаотичної динаміки у цих випадках є особливі рішення. У точках цього рішення порушуються умови теореми єдиності. У чисельних розрахунках це проявляється у тому, що у використовуваних математичних розрахунках з'являється випадковий компонент. Показано, що практично у всіх випадках збільшення точності розрахунків, що проводяться, дозволяє у системах з одним ступенем свободи придушити хаотичну динаміку.