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STOCHASTIC DISTURBANCE BY "WHITE" NOISE WAVE FUNCTIONS IN THE PARABOLIC POTENTIAL AND STATISTICAL WAVE PACKET LIFETIME DISTRIBUTION

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The work examines the parabolic potential, which as the whole is subject to the stochastic influence of the random process of "white" noise, and the dynamics of the wave function of a particle in it. Based on the found solutions to the nonstationary Schrödinger equation, the time evolution of the wave function is considered. The problem of destroying the wave packet of a particle is formulated and analyzed, which is realized when the condition of achieving the dispersion of the packet of a given size is met. For the random variable – time to destruction – the probability distribution density is given.

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INTRODUCTION

The mechanism for changing the properties of materials is associated with the excitation of nonlinear localized vibrations in the lattice [1, 2]. These vibrations, in turn, affect the dynamics of particles in it. The structure and arrangement of atoms in a single crystal significantly influence the nature of the movement of particles in the lattice [3, 4]. The work poses and analyzes the problem of destroying the wave packet of a particle, which is realized when the condition of achieving packet dispersion of a given size is met. For the random variable – time to destruction – the probability distribution density is given.

The quantum mechanical problem of the motion of a particle in the quadratic potential, which as the whole is subject to the stochastic process of "white" noise, is considered. A similar formulation arises in problems when the perturbation represents the trajectory of a one-dimensional or two-dimensional process that models changes in the potential during the movement of a particle, in particular, when an electron moves along the crystal axis [3, 5]. In this case, the role of time in the problem is played not by the depth of particle penetration, but by the perturbation function that describes the forced vibrations of the crystal lattice. Another example is associated with calculating the rate of a chemical reaction near localized anharmonic vibrations of atoms caused by thermal fluctuations or external influences [5, 6]. Here, due to the large amplitude of anharmonic vibrations, the position of the potential well in which the particle is located can no longer be considered stationary, which requires a revision of the problem of calculating the wave function taking into account the dynamics of the potential well.

Below we will consider the time evolution of a particle in the ground state with the initial wave function

$$\psi(x_0,0)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x_0^2\right) \qquad (1)$$

in potential

$$V(x,t)=\frac{1}{2}m\omega^2[x-U(t)]^2, \qquad (2)$$

where $U(t)$ is some square-integrable function. This type of potential is the generalization of the parabolic potential $V(x,t)=m\omega^2x^2/2$ with $U(t)=0$ and, used, in particular, in analyzing the motion of channeling charged particles [3, 4].

EVOLUTION OF THE WAVE FUNCTION IN THE PARABOLIC POTENTIAL

For the wave function $\Psi=\Psi(x,t;x_0,t_0)$, the Schrödinger equation has the form [1, 6, 7]

$$\frac{\hbar}{i}\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{2}m\omega^2[x-U(t)]^2\Psi. \qquad (3)$$

The physical content of the wave function Ψ is that it describes the amplitude of the probability of transition from the state at $t_0=0$ an instant, characterized by the coordinate x_0, to the state at t an instant, characterized by the coordinate x.

Based on the parabolicity of the potential (2), we find that the solution to the Schrödinger equation (3) is as follows [6-8]

$$\Psi(x,t;x_0,t_0)=\left(\frac{me^{i\omega t}}{\pi\hbar(e^{2i\omega t}-1)}\right)^{1/2}\times \qquad (4)$$

$$\exp\left(-\frac{m\omega}{2\hbar}(x^2-x_0^2)-i\frac{m\omega^2}{2\hbar}\int_0^t U^2(\tau)\,d\tau\right)\times$$

$$\exp\left(i\frac{\hbar}{2m}\int_0^t Y^2(\tau)\,d\tau+xY(t)-\right.$$

$$\left.-\frac{m\omega}{\hbar(e^{2i\omega t}-1)}\left[x-e^{i\omega t}x_0-Z(t)\right]^2\right),$$

where

$$Y(\tau)=i\frac{m\omega^2}{\hbar}\int_0^\tau U(\tau')\exp(-i\omega\tau+i\omega\tau')\,d\tau',$$

$$Z(t)=\frac{\hbar e^{i\omega t}}{m}\int_0^t Y(\tau)\exp(-i\omega\tau)\,d\tau.$$

The Gaussian form of expression (4) with respect to the arguments x_0 and x for the resulting function $\Psi(x,t;x_0,t_0)$ (Green's function) follows from the parabolic nature of potential (2). Function $\Psi(x,t;x_0,t_0)$ (4) satisfies the same equation as the wave function $\psi(x_0,0)$, but with the initial condition $\Psi(x,t;x_0,0)=\delta(x-x_0)$.

The value of the wave function $\psi(x,t)$ at the moment $t>t_0$ can be determined from its given value at the moment $t_0=0$:

$$\psi(x,t)=\int_{-\infty}^{\infty}\psi(x_0,0)\,\Psi(x,t;x_0,0)\,dx_0\,. \tag{5}$$

After calculating integral (5) with kernel $\Psi(x,t;x_0,0)$ (4) and initial wave function $\psi(x_0,0)$ (1), we obtain

$$\psi(x,t)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-i\frac{m\omega^2}{2\hbar}J-\frac{m\omega}{2\hbar}\left[x-X(t)\right]^2\right), \tag{6}$$

$$X(t)=\omega\int_0^t U(\tau)\sin(\omega t-\omega\tau)\,d\tau,$$

where J is the phase shift of the resulting wave function

$$J=\int_0^t U^2(\tau)d\tau+\frac{t\hbar}{m\omega}+2x\int_0^t U(\tau)\cos\big(\omega(t-\tau)\big)d\tau+$$

$$+\int_0^t d\tau\int_0^t d\tau' U(\tau)U(\tau')\sin\big(\omega(\tau-t)\big)\cos\big(\omega(\tau'-t)\big). \tag{7}$$

If at $t=0$ the particle was in the ground state with wave function (1), then at the moment of time we will find for the probability density $p(x,t)$

$$p(x,t)=\left|\psi(x,t)\right|^2=\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{m\omega}{\hbar}\left[x-X(t)\right]^2\right)\cdot \tag{8}$$

Thus, if a particle experiences a time evolution in potential (2) with $U(t)=0$, then by the moment of time t the probability distribution $p(x,t)$ remains the same as at the moment $t_0=0$. The state of the wave packet changes significantly if $U(t)\neq 0$, as can be seen from the characteristics of the wave packet.

Next, we will consider the case when $U(t)$ is a stochastic "white" noise process with zero mathematical expectation and variance $\sigma_{\rm U}^2$. In Fig. 1 shows examples of density profiles $p(x,t)$ obtained based on solving equation (3). If the modulation function $U(t)$ contains a stochastic component, then $\langle x(t)\rangle$ and $\langle x^2(t)\rangle$ are also time-dependent random variables. Here and below, angle brackets $\langle\ldots\rangle$ indicate the operation of finding the mathematical expectation relative to the realizations of the stochastic process of "white" noise $U(t)$.

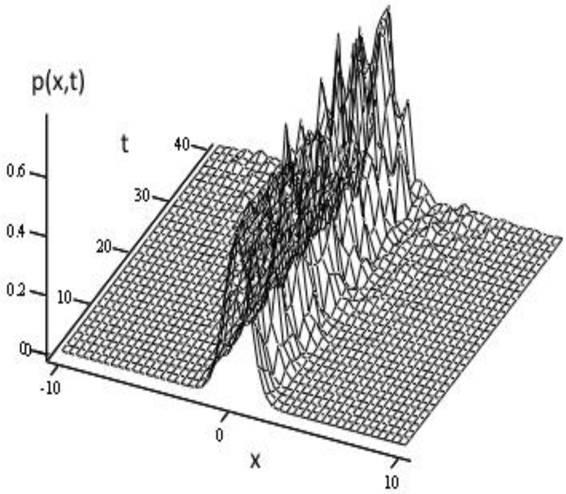

Fig. 1. An example of the evolution of the density $p(x,t)$ of wave packet perturbed by the process of "white" noise; parameters: $\omega=1$, $\sigma_{\rm U}=1$

Over time t, the density $p(x,t)$ undergoes transformations reflecting the influence of the function $U(\tau)$. Quantum mechanical averages – the first and second moments of the probability distribution density are:

$$\langle x(t)\rangle=\left\langle \omega\int_0^t U(\tau)\sin(\omega t-\omega\tau)\,d\tau\right\rangle=0\,. \tag{9}$$

For the second moment we write

$$\langle x^2(t)\rangle=\frac{\hbar}{2m\omega}+\omega^2\times \tag{10}$$

$$\left\langle\int_0^t\int_0^t U(\tau)U(\tau')\sin(\omega t-\omega\tau)\sin(\omega t-\omega\tau')\,d\tau\,d\tau'\right\rangle.$$

Due to the properties of "white" noise, $\langle U(\tau)U(\tau')\rangle=\sigma_{\rm U}^2\delta(\tau-\tau')$ therefore

$$\langle x^2(t)\rangle=\frac{\hbar}{2m\omega}+\omega^2\sigma_{\rm U}^2\int_0^t\sin^2(\omega t-\omega\tau)\,d\tau\,. \tag{11}$$

For sufficiently large times, $t>>\omega^{-1}$, we have

$$\langle x^2(t)\rangle=\frac{\hbar}{2m\omega}+\frac{\omega^2\sigma_{\rm U}^2}{2}t\,. \tag{12}$$

WAVE PACKET DESTRUCTION AND DENSITY OF DISTRIBUTION TIMELIFE

Now we can formulate the problem statement. Let us set the horizontal size H and let it be given that upon fulfillment of the condition

$$x^2(t)=H^2 \tag{13}$$

the particle drops out of consideration (dies).

Let's denote $\eta(\tau)=x^2(\tau)$ Since $\eta(\tau)$ it is a random process, the moment t of reaching the level will also be random; it is uniquely determined by the level of achievement $\eta(t) = H^2$ due to the monotonic increase $\eta(\tau)$. So, a random event $\{A: x^2(t) = H^2\}$ will take place at a random moment in time Т. Let us pose the problem of the probability distribution function $F(t)$ of a random variable – *lifetime* T – and its statistical characteristics.

The distribution density $f(t)$ of random variable T values can be written as

$$f(t) = \left\langle \delta\left(t - \eta^{-1}(H^2)\right)\right\rangle. \quad (14)$$

From the type of event $\{A: x^2(t) = H^2\}$ and properties of the δ-function it follows

$$f(t) = \left\langle \delta\left(x^2(t) - H^2\right)\left(\frac{d}{dt}x^2(t)\right)\right\rangle. \quad (15)$$

Here you can use the integral representation for the δ-function:

$$f(t) = \frac{1}{2\pi}\frac{d}{dt}\left\langle \int_{-\infty}^{\infty} \exp\left[i\lambda\left(x^2(t) - H^2\right)\right]\frac{d\lambda}{i\lambda}\right\rangle. \quad (16)$$

For a given moment in time t, finding the mathematical expectation in (15) is carried out by integrating over the set { x } of realizations of the random process $x(t)$ at this moment

$$f(t) = \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty}\frac{d\lambda}{i\lambda}\int_{-\infty}^{\infty} dx\,\rho(x)\exp\left[i\lambda\left(x^2 - H^2\right)\right] \quad (17)$$

with density $\rho(x)$ and dispersion D_t according to (9) and (12):

$$\rho(x) = \frac{1}{\sqrt{2\pi D_t}}\exp\left(-\frac{x^2}{2D_t}\right), \quad (18)$$

$$D_t = \frac{\hbar}{2m\omega} + \frac{\omega^2\sigma_{\mathrm{U}}^2}{2}t. \quad (19)$$

Integrating into (17) over x , we write

$$f(t) = \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty}\frac{d\lambda}{i\lambda\sqrt{1 - 2i\lambda D_t}}\exp\left(-i\lambda H^2\right), \quad (20)$$

or after replacement $z = \lambda D_t$:

$$f(t) = \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty}\frac{dz}{iz\sqrt{1 - 2iz}}\exp\left(-iz\frac{H^2}{D_t}\right). \quad (21)$$

Calculation of the derivative and subsequent replacement of the integration variable $z = (1 - \xi^2)/2i$ gives for the density

$$f(t) = \frac{\omega^2\sigma_{\mathrm{U}}^2 H^2}{4\pi i D_t^2}\exp\left(-\frac{H^2}{2D_t}\right)\int_C d\xi\,\exp\left(\frac{\xi^2 H^2}{2D_t}\right). \quad (22)$$

After replacing the integration variable and choosing the Bromwich contour *C*, we will find integration for ξ the desired probability density distribution $f(t)$ of the random variable Т – the lifetime of the wave packet under stochastic disturbance (Fig. 2):

$$f(t) = \frac{\omega^2\sigma_{\mathrm{U}}^2}{2\sqrt{2\pi}}\frac{H}{D_t^{3/2}}\exp\left(-\frac{H^2}{2D_t}\right). \quad (23)$$

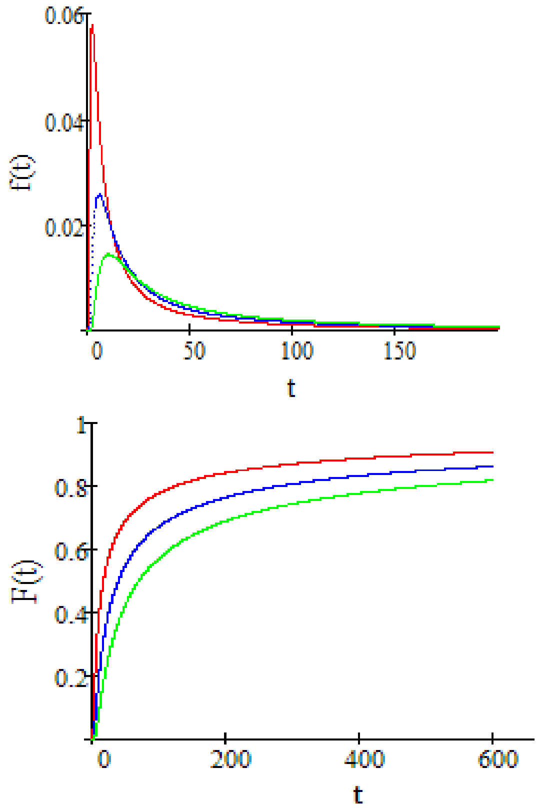

Fig. 2. Distribution density $f(t)$ (top) and integral probability $F(t)$ (bottom) of the lifetime Т*; parameters: $\omega = 1$, $\sigma_{\mathrm{U}} = 1$; H=2 (red), H=3 (blue), H=4 (green)*

The destruction of the wave packet occurs at a sufficiently large dispersion value D_t. Then the contribution of the ground state of the wave function can be neglected. In this case

$$f(t) = \frac{H}{\sqrt{\pi}\,\omega\sigma_{\mathrm{U}}t^{3/2}}\exp\left(-\frac{H^2}{\omega^2\sigma_{\mathrm{U}}^2 t}\right), \quad (24)$$

or in a more compact notation

$$f(t) = \sqrt{\frac{<t>}{\pi t^3}}\exp\left(-\frac{<t>}{t}\right), \quad <t> = \frac{H^2}{\omega^2\sigma_{\mathrm{U}}^2}. \quad (25)$$

Density (24) is normalized to unity. Density $f(t)$ has no statistical moments, and the value $<t>$ in (25) can be interpreted as the “average time” of life. The dependences of density $f(t)$ and probability of destruction $F(t)$ by time t are shown in Fig. 2.

From formula (24) it can be seen that as the horizontal size H increases, the probability of destruction decreases, but at the same time this probability increases with increasing intensity σ_{U}^2 of “white” noise.

CONCLUSIONS

The work considers the parabolic potential, which as a whole is subject to stochastic influences such as "white" noise and the dynamics of the wave function of the particle in it. Based on the nonstationary Schrödinger equation, analytical expressions for the dynamics of the wave function are constructed. The evolution of the particle wave function is analyzed. The problem of the lifetime of a wave packet is considered, which is perturbed by the stochastic process of "white" noise and is destroyed under the condition that its dispersion has reached a given size and the particle is eliminated from consideration (perishes). Due to the stochastic nature of the disturbing process, the time interval before destruction will also turn out to be random. An analytical expression is obtained for the distribution density of the random variable – lifetime.

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СТОХАСТИЧНЕ РОЗКАЧУВАННЯ «БІЛИМ» ШУМОМ ХВИЛЬОВОЇ ФУНКЦІЇ У ПАРАБОЛІЧНОМУ ПОТЕНЦІАЛІ І СТАТИСТИЧНИЙ РОЗПОДІЛ ЧАСУ ЖИТТЯ ХВИЛЬОВОГО ПАКЕТА

О.С. Мазманішвілі

Розглянуто параболічний потенціал, який як ціле схильний до стохастичного впливу випадкового процесу «білого» шуму, і динаміка хвильової функції частинки в ньому. На основі знайдених рішень нестаціонарного рівняння Шредінгера розглянуто часову еволюцію хвильової функції. Поставлена та проаналізована задача руйнування хвильового пакета частинки, яке реалізується при виконанні умови досягнення дисперсії пакета заданому розміру. Для випадкової величини – часу до руйнування – наведено щільність розподілу ймовірностей.