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NUMERICAL TREATMENT OF LOWER HYBRID RESONANCE IN PLASMA

Yu.M. Marchuk[1], V.E. Moiseenko[1,2], Yu.S. Kulyk[1], T. Wauters[3]

National Science Center "Kharkiv Institute of Physics and Technology", Institute of Plasma Physics, Kharkiv, Ukraine;
[2]Ångström Laboratory, Uppsala University, Uppsala, Sweden;
[3]ITER Organization, St. Paul-lez-Durance, France

E-mail: YuriiiMarchuk@gmail.com

In this work, the slow wave equation for the E_z component and the expression for the Poyting vector were obtained from the system of Maxwell's equations. The analysis of these equations is given with respect of the lower hybrid resonance point presence. The capacities of the penalty method and the analytical continuation method are analyzed in connection to their usage for electromagnetic plasma wave propagation numerical treatment using the Galerkin approach within finite element discretization. Numerical calculations of the slow wave equation were carried out, values of relative errors were calculated for the two methods, and the results were analyzed.

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INTRODUCTION

Waves propagating in plasma in the ion cyclotron range of frequencies (ICRF) are the fast magnetosonic wave and slow wave (SV). SV that propagates in a homogeneous or weakly non-uniform plasma characterized by the dispersion equation [1] (1):

$$k_\perp^2 = -\frac{\varepsilon_\parallel}{\varepsilon_\perp}\left(k_\parallel^2 - k_0^2\varepsilon_\perp\right), \qquad (1)$$

where $k_\perp^2 = k^2 - k_\parallel^2$, $k_\parallel = \mathbf{b}\cdot\mathbf{k}\cdot\mathbf{b}$, $\mathbf{b} = \mathbf{B}/B$, $k_0 = \omega/c$, the dielectric tensor has a form [4]:

$$\hat{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon_\perp & ig & 0 \\ -ig & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_\parallel \end{pmatrix}.$$

In an inhomogeneous plasma, the condition for WKB approximation are fulfilled, if the inhomogeneity of plasma is small and the wavenumber varies weakly: $|\nabla\cdot\mathbf{k}| \ll |\mathbf{k}^2|$. The Cartesian geometry is considered here, and the magnetic field is directed along z-axis. Plasma is uniform along y and z-axes and non-uniform along the x axis.

From the SV dispersion equation (1) we see that in the case of $\varepsilon_\perp \to 0$ $k_\perp \to \infty$ which indicates appearance of the lower hybrid resonance (LHR), see, e.g. [1-6]. If the plasma has more than one ionic components, then LHR can appear at frequencies between the cyclotron frequencies of the ions, and LHR in this case is also called an ion-ion hybrid resonance. The WKB approximation gives faster and faster oscillations on approach the hybrid resonance point, but WKB approximation fails to reproduce the fields at LHR vicinity: in the hybrid resonance itself the fields asymptotic behavior is: $E_x = -i\frac{C}{k_\parallel x}$ and $E_z = C\ln(x)$ (see, e.g. [2, 3]). Both electric field components are singular.

Numerical methods reconstruct the approximate solutions with regular basis functions, which can be polynomials or sine and cosine functions, and, because of this, it is impossible to describes the singular solution within the framework of standard numerical methods. Meanwhile, the case of LHR is practical and needs numerical treatment. The main characteristic of the lower hybrid resonance is the power that enters it at is absorbed in it. We will not focus on reproducing all the electric fields in this lower hybrid resonance, but will limit ourselves to calculating the absorbed power. Therefore, the formulation of the problem in this work is as follows: to examine different possibilities to calculate the absorbed power in the lower hybrid resonance within the numerical approach.

In the model, along the x axis non-uniform plasma placed in a uniform magnetic field directed along the z axis is considered. Such a model reflects all the principal features of the lower hybrid resonance. In further algebra we extract the equation for the slow wave from the time-harmonic Maxwell equation, taking into account that the slow wave has a fairly short wavelength and the weak coupling between the fast and slow waves can be neglected.

EQUATION FOR SLOW WAVE

The time-harmonic Maxwell's equations are considered in the form [4] (2):

$$\nabla\times\nabla\times\boldsymbol{E} - \frac{\omega^2}{c^2}\hat{\boldsymbol{\varepsilon}}(x)\cdot\boldsymbol{E} = i\omega\mu_0\boldsymbol{j}, \qquad (2)$$

where **E** is the temporal Fourier harmonic of the electric field and **j** is the density of the external RF (antenna) electric current. The plasma dielectric tensor is a function of the x – coordinate due to the plasma density non-uniformity.

The small parameter for the problem appears from the fact that the x – component of wavenumber is dominant. Considering that $k_x^2 \gg k_y^2, k_z^2$, the vector components of the equations take the following form. Projections of (2) on the coordinate axes are (3)-(5):

$$\left(k_z^2 + k_y^2 - k_0^2\varepsilon_\perp\right)E_x + ik_y\frac{d}{dx}E_y + \\ +ik_z\frac{d}{dx}E_z - ik_0^2 g E_y = i\omega\mu_0 j_x, \quad (3)$$

$$ik_0^2 g E_x + ik_y\frac{d}{dx}E_x - \frac{d^2}{dx^2}E_y - \\ -k_0^2\varepsilon_\perp E_y - k_y k_z E_z = i\omega\mu_0 j_y, \quad (4)$$

$$ik_z\frac{d}{dx}E_x - k_z k_y E_y - \frac{d^2}{dx^2}E_z + \\ +k_y^2 E_z - \varepsilon_\parallel k_0^2 E_z = i\omega\mu_0 j_z. \quad (5)$$

The terms marked in color are much smaller than the main terms in the slow wave equation, so they can be omitted. Once we have done this, we can perform the integrations in equation (4) and substitute the result into formula (3). Using the modified equation (3) we find expression for the field E_x and calculate its derivative with respect to x. After this, we obtain the final form of the equation SV:

$$\frac{d}{dx}\left\{\left(\frac{k_0^2\varepsilon_\perp}{k_z^2 - k_0^2\varepsilon_\perp}\right)\frac{d}{dx}E_z\right\} + \varepsilon_\parallel k_0^2 E_z = \frac{i\omega^2\rho\mu_0 k_z}{k_z^2 - k_0^2\varepsilon_\perp} - \\ i\omega\mu_0 j_z\left(\frac{k_0^2\varepsilon_\perp}{k_z^2 - k_0^2\varepsilon_\perp}\right). \quad (6)$$

Using the expression: $\nabla\cdot\mathbf{J} - i\omega\rho = 0$ we had modified the expressions for the right hand side of (6). The original system, after omitting the terms that are small, did not change the order with respect to differentiating in x. It remained of fourth order, but the final equation has the second order. The order of the system was decreased to the second one by integration. The appeared integration constants are attributed to the fast wave, and they are dropped.

The equation (6) is accurate when the excitation currents lie outside the plasma region or wherever $\varepsilon_\perp \approx 1$. The energy flow must be irreplaceable, for this we will obtain the equation for the Poyting vector (7)-(9):

$$\Pi_x = \frac{c}{8\pi}(\boldsymbol{E}^* \times \boldsymbol{B})_x = -\frac{c}{8\pi}\left(E_z^* B_y\right), \quad (7)$$

$\boldsymbol{B}$ is the SV magnetic field.

$$E_z^* B_y = \frac{E_z^*(x)}{i\omega}\left(\frac{k_0^2\varepsilon_\perp}{k_z^2 - k_0^2\varepsilon_\perp}\right)\frac{d}{dx}E_z. \quad (8)$$

And, finally

$$\Pi_x = -\frac{1}{8\pi i}\frac{c}{\omega}E_z^*(x)\left(\frac{k_0^2\varepsilon_\perp}{k_z^2 - k_0^2\varepsilon_\perp}\right)\frac{d}{dx}E_z(x). \quad (9)$$

NUMERICAL APPROACH

For numerical exercises we use the following equation (10) which is a specific case of Eq. (6) and describing the LHR phenomenon.

$$\frac{d}{dx}F\frac{d}{dx}E_z + GE_z = 0, \quad (10)$$

$$F = \frac{n_\parallel^2\varepsilon_\perp}{n_\parallel^2 - \varepsilon_\perp}, \quad (11)$$

$$G = k_\parallel^2 L^2(-\varepsilon_\parallel). \quad (12)$$

We use dimensionless coordinate $x \to x/L$, x=0…1, $n_\parallel^2 = k_\parallel^2/k_0^2$, further $n_\parallel^2 - \varepsilon_\perp \approx n_\parallel^2$. In the calculation series below we set $\varepsilon_\perp = -x_r + x$, $G = G_0(1 + x_r - x)$, where x_r is the LHR coordinate, G_0 is a constant. The boundary conditions $E_z(x=0) = 0$ and $E_z(x=1) = 1$ are used. One of the practical cases covered by the Eq. (19) with such parameters is: $\omega = 3\cdot10^8\ \mathrm{c}^{-1}$, $n(x_r) = 10^{17}\ \mathrm{m}^{-3}$, deuterium ions, B = 2.5 T, $k_\parallel = 3\ \mathrm{m}^{-1}$, L = 0.05 m.

Electric field E_z have a singularity at the LHR point, and it is impossible to do a numerical treatment by standard methods to carry out a description of this field. Here we consider two methods of avoiding this problem, first the Penalty method.

PENALTY METHOD

If you look at the function F, its nominator contains $\varepsilon_\perp$, which turns to zero at LHR and leads to a singularity in Eq. (19) solutions. To avoid this, a complex additive (patch) is added to F: $F^* = F + F_{im}$ in a local area $|F| < F_0$ and, as a result, the complex number modulus never vanishes and accordingly there is no numerical singularity, and standard methods of discretization of the equation can be applied.

In first series of calculations the following formula for the linear patch is used (13):

$$F_{im} = i\sqrt{F_0^2 - F^2}. \quad (13)$$

Parabolic patching is a bit more complicated (14):

$$F_{im} = i\sqrt{\left(\frac{F^2}{2F_0} + \frac{F_0}{2}\right)^2 - F^2}. \quad (14)$$

ANALYTICAL CONTINUATION METHOD

The second method is analytical continuation. The essence of the method is that if we have a system of Maxwell's equations in components and this system does contain only coefficients that are analytical functions of the coordinate, then we should expect a solution of this system to also be an analytical function. If this is so, then the solution found on the line of real x will be extended to the complex plane $x \to p = x + is$.

For using this feature, the idea is to modify the path of integration, and now at the complex plane around the singularity we circumvent the LHR point making the path that goes around it. It is expected that the solution of the system will not be singular, but will be a smooth function. Then at this path it is possible to apply standard integration methods, for example, the Galerkin method.

In this way, we get that solution not at the point itself, but near it and this solution differs from what would be, but does not contain a singularity. The analytical continuation could be made back to the real axis, but this is not practical since we would get a singularity.

We have an analytical continuation, and we must choose a path and go to the complex plane. After that, we will see how the choice of a complex path will affect the calculation results.

The bypass path is defined as follows $p = x+i\lambda(x)$. The first choice is a parabolic bypass function (15):

$$\lambda(x) = \frac{x_w^2-(x-x_s)^2}{x_w}, \quad (15)$$

where x_w is width of excursion area in x coordinate, x_s is position of a circumvented (LHR) point. In the formulas for the Galerkin method, in addition to λ, there is a derivative, which has the form (16):

$$\frac{d}{dx}\lambda(x) = -\frac{2(x-x_s)}{x_w}. \quad (16)$$

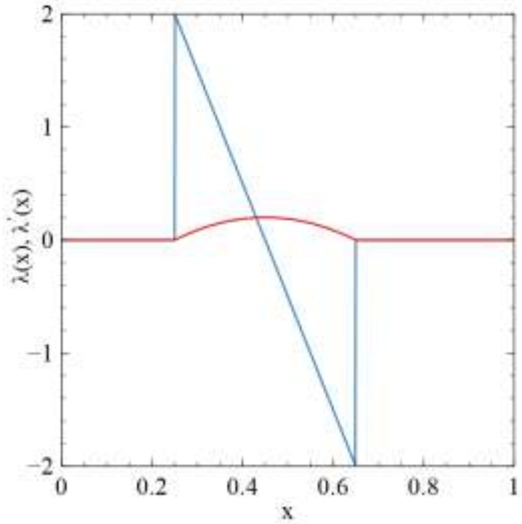

Fig. 1. Patching vs. coordinate. The red line is λ(x) and the blue line $\frac{d}{dx}$λ(x) for parabolic bypass function

As we can see from the Fig. 1, the derivative of the parabolic patch undergoes a jump at initial and end points.

Another approach to forming a bypass using the cosine function is (17) and (18):

$$\lambda(x) = x_w cos^2\left[\frac{\pi(x-x_s)}{2x_w}\right], \quad (17)$$

$$\frac{d}{dx}\lambda(x) = -\frac{\pi}{2}sin\left[\frac{\pi(x-x_s)}{x_w}\right]. \quad (18)$$

GALERKIN METHOD

We use the Galerkin method (see e.g. [7]) with first-order finite elements (FE). The formula for the finite element is:

$$\Lambda_i(x) = \begin{cases} 0, & x < x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} < x < x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i < x < x_{i+1} \\ 0, & x > x_{i+1} \end{cases}$$

Any smooth function can be approximated as a finite element expansion (19) and (20).

$$E_z(x) \approx \sum_i E_{zi}\,\Lambda_i(x), \quad (19)$$

$$\int_x \Lambda_{n'}(x)\left[\frac{d}{dx}F\frac{d}{dx}E_z + GE_z\right]dx = \mathrm{R}. \quad (20)$$

The right-hand side R of the equation is determined by the boundary conditions. This system is computationally solved with the help of LU decomposition (Fig. 2).

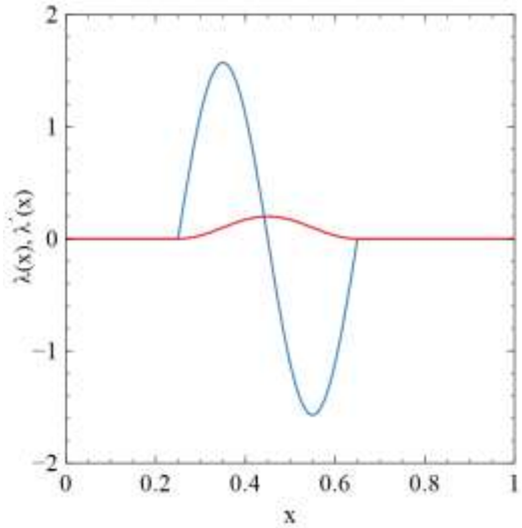

Fig. 2. Patching vs. coordinate. The red line is λ(x) and the blue line $\frac{d}{dx}$ λ(x) for cosine bypass function

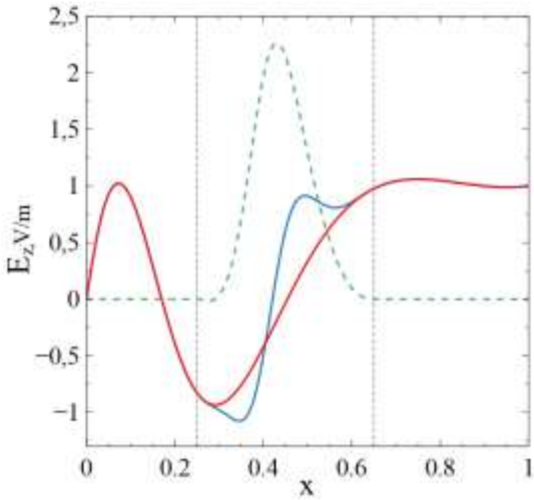

Fig. 3. Electric field E_z vs. coordinate. The red line is the value of the field without entering the complex plane. The blue line is the value of the field when entering the complex plane. The green dotted line is an imaginary part. The vertical dashed lines are the width of the bypass zone

RESULTS OF CALCULATIONS

VERIFICATION FOR ANALYTICAL CONTINUATION METHOD

Verification of the analytical continuation method is made by examining the case with no singularities [4]. Bypassing is performed around a point, the solutions at which are not singular. It is expected that we should obtain coinciding solutions at the real axis for the cases of non-bypassing and bypassing. Numerical experiments was done for G_0 = 100 and x_r = -0.15.

Fig. 3 illustrates divergence of solutions in the circumventing area and coincidence of them at the areas behind.

The relative error of E_z approximation $\delta = \sqrt{\frac{\sum_i |E_{zi} - E_z(x_i)|^2}{\sum_i |E_{zi}|^2}}$ is calculated at the areas without excursions of integration path to the complex plane, in our case $x \in (0\ldots0.25;\ 0.65\ldots1)$. The solution obtained for N=10000 is assumed to be very close to the exact one and used instead of it. As we can see in Fig. 4, both parabolic and cosine bypasses show convergence corresponding to the approximation of the discretization method. The cosine bypass function gives better results likely to the fact that this function does not have a break of derivative over x at the end points.

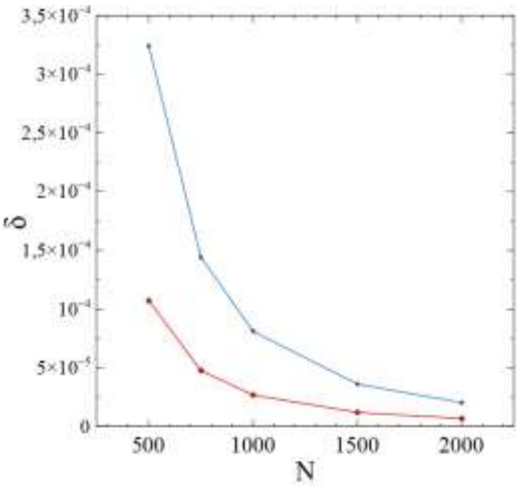

Fig. 4. Relative error δ in E_z vs. number N of mesh nodes. The red line is for parabola bypass function, the blue line is for cosine bypass function

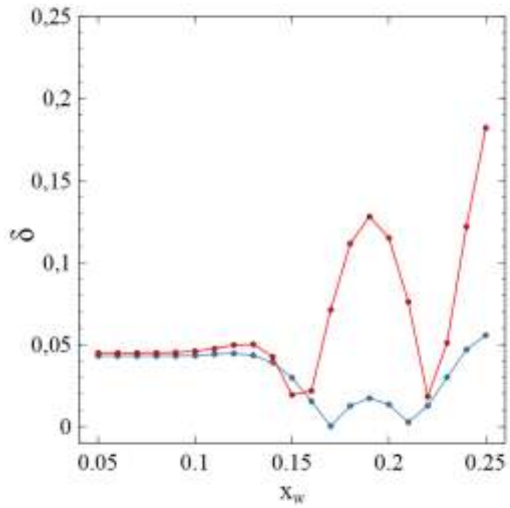

Fig. 5. δ (relative error of energy flux) vs. x_w bypass width. The red line is for linear patch (10), blue line is for parabolic patch (11)

NUMERICAL TREATMENT OF PENALTY METHOD

Test calculations were performed for the case when the resonance point is inside the calculation domain when G_0=100 and x_r=0.45. In the previous case there was no absorption, the energy flux was zero at all points. Therefore, the energy flux was not used for the assessment. In this case, due to the presence of a local hybrid resonance point, there is power absorption in it and, therefore, the energy flux is not zero to the right of the LHR point. The flux is chosen as a criterion for the correctness of the solution and the relative error was calculated against it. As before, we substitute the exact value of the flux by the calculated one at N=10000,

We analyzed the Penalty method for two complex patches (10) and (11).

We see from Fig. 5 that for the red curve (linear patch) for x_w values greater than x_w=0.17 the error becomes large, and for smaller x_w values it is lesser. We can conclude that the effect of LHR is described well but not with high accuracy. In case of usage of parabolic patch we see that the relative error for wide range of the patch width is better than for linear patch, below 5%, which is probably a result of smoother shape of this patch.

NUMERICAL EXPERIMENTS ON ANALYTICAL CONTINUATION METHOD

We have investigated two bypasses and in two cases the solution error is within the grid approximation. In a large interval from for x_w=0.05 to 0.25 δ do not go over the error limits of 10^{-4} (Fig. 6). For the cosine bypass the error is smaller presumably because the derivative of the bypass function is continuous at the bypass end points, while for the parabolic bypass the derivative jumps, see Figs.1 and 2.

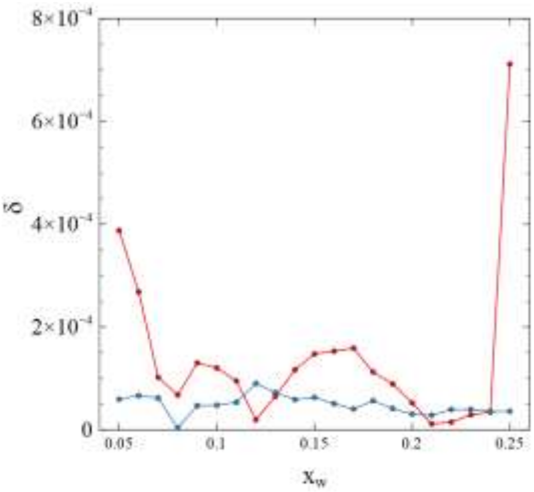

Fig. 6. δ (relative error of energy flux) vs. x_w bypass width. Red line for parabolic, blue line for cosine

SUMMARY AND CONCLUSIONS

Near the point of LHR, the conditions for which $\varepsilon_\perp = 0$, the slow wave field undergoes a discontinuity while the fast wave field has a regular point. In order to conduct a study of the slow wave behavior, we obtained an equation for the slow wave for the field component E_z. This equation was solved numerically using the Galerkin method. Since the Galerkin method with finite elements fails to reproduce the singularity at the LHR point, two possibilities for obtaining a solution in this case were used. This article discusses the results of numerical experiments with the SW equations using the Galerkin method and first-order finite elements.

The singularity in the equation solution is caused by the zeroing of the coefficient at the senior derivative.

To avoid singularity, in frame of the penalty method, one can use an artificial small complex patch to this coefficient, and then the module of it will never be zeroed. In this work we used linear and parabolic patches. In a wide range the patch sizes, the absorbed power in the LHR is calculated satisfactorily. The relative error of determining the absorbed power is within 17% for a linear patch. For a parabolic patch, this accuracy is higher and the relative error is below 5%. The Penalty method is relatively simple to use and in many cases leads to a satisfactory result in calculating the absorbed power. It is actually possible to achieve an accuracy of 5%, which in many cases may be sufficient.

The approach of analytical continuation is based on the fact that if in the equation for LHR there are only the analytical functions, then the solutions will also be analytical functions. This is the basis for putting the integration path near the LHR region to complex plane and thereby avoiding the appearance of a singularities in the solutions. The numerical experiments shown that using this approach it is possible to obtain the accuracy of calculating the absorbed power significantly better than with the penalty method, and the accuracy of these calculations agrees with the approximation of the numerical model. Two methods of bypassing were used - parabolic and cosine and both methods give good accuracy of calculations, but the results are significantly better for the cosine bypass function since there is no discontinuity of the bypass function derivative at the edges of the bypass area.

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ЧИСЛОВЕ МОДЕЛЮВАННЯ НИЖНЬОГО ГІБРИДНОГО РЕЗОНАНСУ В ПЛАЗМІ

Ю.М. Марчук, В.Є. Моісеєнко, Ю.С. Кулик, T. Wauters

З системи рівнянь Максвелла отримано рівняння повільної хвилі для компоненти E_z та вираз для вектора Пойтінга. Аналіз цих рівнянь наведено з огляду на наявність нижньої гібридної точки резонансу. Проаналізовано можливості методу штрафу та методу аналітичного продовження у зв'язку з їх використанням для числового моделювання поширення електромагнітної плазмової хвилі з використанням підходу Галеркіна в рамках скінчено-елементної дискретизації. Проведено числові розрахунки рівняння повільної хвилі, розраховано значення відносних похибок для двох вищезгаданих методів та проаналізовано результати.