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ANALYTICAL SOLUTIONS OF THE THERMODYNAMIC POTENTIAL OF THE MICHELSON MODEL IN EQUILIBRIUM SYSTEMS

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The paper investigates exact solutions of a nonlinear fourth-order differential equation that follows from the variational principle for a thermodynamic potential with high derivatives. It is shown that certain classes of solutions can be represented by elliptic Jacobi functions of the delta-amplitude and elliptic cosine types, and in limiting cases – by localised Bell solitons. The conditions for the existence of real solutions are analyzed, and a relationship between the parameters of the equation and phase transitions to modulated states is established. In particular, the critical temperatures of second-order transitions corresponding to the emergence of modulated phases (MS1, MS2) are determined, and restrictions on the parameters of the thermodynamic potential are given. The results obtained are consistent with previous works and deepen the understanding of equilibrium states in systems with strong nonlinearity.

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INTRODUCTION

Phase transitions in systems with complex structures and high derivatives in thermodynamic potential attract considerable attention in modern condensed matter physics [1–6]. They describe a wide range of phenomena, from structural transformations in crystals to the formation of modulated and quasi-periodic phases.

The presence of impurities or defects often leads to blurring of phase boundaries, the appearance of metastable phases, or a decrease in phase transition temperatures. On the other hand, in highly pure samples, it is possible to observe new types of phase transitions that cannot be observed in technical materials. For example, article [7] describes a paramagnetic-diamagnetic transition in high-purity metallic ytterbium, accompanied by an fcc–hcp structural transformation, where a first-order transition uncharacteristic of ytterbium with impurities was recorded for the first time.

Of particular interest is the problem of electric charge relaxation in dielectric materials, which is closely related to the mechanisms of phase transitions [8, 9]. In such systems, a change in the interphase lattice is accompanied by a restructuring of local electric fields and a change in the nature of conductivity. The study of charge dynamics allows for a deeper understanding of nonlinear processes and the establishment of conditions for the emergence of new phase states.

The study of such transitions requires not only numerical modelling, but also the search for exact or partially exact solutions to the corresponding variational equations, which allows the characteristic properties of the system to be identified. This work considers the use of Jacobi elliptic functions [10–12] to find solutions to a nonlinear fourth-order differential equation that follows from the variational principle for a thermodynamic potential with additional terms of higher derivatives. The aim is to construct exact solutions for the order parameter, analyze their physical interpretation, and determine the conditions under which phase transitions to modulated states occur.

CALCULATION

We will consider thermodynamic potential in the form [12]:

$$F = F_0 + \int dx \left[\frac{1}{2}(\varphi'')^2 - \frac{g}{2}(\varphi\varphi')^2 - \frac{\gamma}{2}(\varphi')^2 + \frac{q}{2}\varphi^2 + \frac{p}{4}\varphi^4 + \frac{1}{6}\varphi^6\right], \qquad (1)$$

where g, γ are material parameters and $q = q_0(T - T_c)/T_c$ and p depend on other conditions (e.g. pressure).

The variational equation for functional (1) is the following nonlinear fourth-order differential equation:

$$\varphi^{(IV)} + g\left[\varphi^2\varphi'' + \varphi\varphi'^2\right] + \gamma\varphi'' + q\varphi + p\varphi^3 + \varphi^5 = 0. \qquad (2)$$

Let us represent the order parameter as

$$\varphi(x) = a * dn(bx, k). \qquad (3)$$

Substituting (3) into (2), we obtain the following equations

$$a^4 - 3ga^2b^2 + 24b^4 = 0; \qquad (4)$$

$$b^4(20k^2 - 40) + 2ga^2b^2(2 - k^2) - 2\gamma b^2 + pa^2 = 0; \qquad (5)$$

$$b^4(k^4 - 16k^2 + 16) + ga^2b^2(k^2 - 1) + \gamma b^2(2 - k^2) + q = 0. \qquad (6)$$

From equation (4), we find the amplitude a of the order parameter:

$$a^2 = 3\frac{g \pm \sqrt{g^2 - \frac{32}{3}}}{2} b^2 = \acute{a}_{\pm}{}^2 b^2. \quad (7)$$

We see that parameter a is valid only when

$$g \geq \sqrt{\frac{32}{3}}. \quad (8)$$

Let us substitute (7) into equation (5):

$$b^2 = \frac{\xi\sigma}{2(2-k^2)}, \quad (9)$$

where

$$\xi = 2\gamma - p\acute{a}^2, \quad (10)$$

$$\frac{1}{\sigma} = g\acute{a}^2 - 10. \quad (11)$$

Substituting (9) into equation (6), we obtain the following expression for the delta amplitude modulus (3):

$$\alpha k^4 - \beta k^2 + \beta = 0, \quad (12)$$

where

$$\alpha = \sigma^2\xi^2 + 2\gamma\sigma\xi + 4q, \quad (13)$$

$$\beta = (6\sigma^2 - \sigma)\xi^2 + 8\gamma\sigma\xi + 16q. \quad (14)$$

The final expression for the order parameter module from (12):

$$k^2 = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\beta}}{2\alpha}. \quad (15)$$

Let us consider the limiting case when $k^2 = 0$. From equation (12) we obtain

$$\beta = 0 \quad (16)$$

or

$$q_0 = -\frac{\gamma\sigma\xi}{2} - \frac{(6\sigma^2 - \sigma)\xi^2}{16}. \quad (17)$$

That is, at temperature (17), a second-order phase transition to a modulated phase occurs.

Now let us consider the limiting case when the parameter $k^2 = 1$. From equation (12), we see that for

$$\alpha = 0 \quad (18)$$

or

$$q_1 = -\frac{\gamma\sigma\xi}{2} - \frac{\sigma^2\xi^2}{4}. \quad (19)$$

At temperature (19), the order parameter (3) will appear as a bell soliton.

Depending on the values of σ, the value of the parameter q (17) and (19), which depends on temperature, will differ.

If $\sigma \in [0,\ 1/2]$, then $q_0 > q_1$, for other values of $\sigma - q_0 < q_1$.

In [10], it was shown that to describe the sequence of phase transitions: highly symmetric-modulated-low symmetric phases, the solution (2) should be sought in the form

$$\varphi'^2(x) = C + B\varphi^2 - A\varphi^4, \qquad A > 0. \quad (20)$$

The solutions (20) are the Jacobian functions (3) and

$$\varphi(x) = a * cn(bx, k). \quad (21)$$

This circumstance emphasises that functions (3) and (21) are partial exact solutions (20). At the same time, there is no certainty that these partial solutions are absolute extrema of the thermodynamic potential (1) corresponding to the equilibrium states of the system. The obtained function (20) will not identically transform the variational equation (2) to zero. However, it can be considered that it is closer to the absolute extremum of the functional (1) than the solutions found by direct substitution of the Jacobi functions (3) or (21) into equation (2).

Substituting (20) into (2), we obtain the following system of equations

$$\begin{cases} 24A^2 - 3gA + 1 = 0 \\ -20BA + 2gB - 2\gamma A + p = 0 \\ B^2 - 12AC + gC + \gamma B + q = 0 \end{cases}. \quad (22)$$

From the first equation (22) we get:

$$A = \frac{g \pm \sqrt{g^2 - \frac{32}{3}}}{16} = \frac{1}{\acute{a}_{\pm}{}^2}. \quad (23)$$

From the second equation (22) and substituting (23), we find:

$$B = \frac{2\gamma A - p}{2(g - 10A)} = \frac{\xi\sigma}{2}. \quad (24)$$

Finally, we obtain parameter C by substituting (23) and (24) into the third equation (22):

$$C = \frac{B^2 + \gamma B + q}{12A - g} = \acute{a}_{\pm}{}^2\sigma\frac{\sigma^2\xi^2 + 2\gamma\sigma\xi + 4q}{4(2\sigma - 1)}. \quad (25)$$

Let us consider the evolution of the phase. We will assume that the parameter $\sigma > 0$, and accordingly $\xi < 0$.

1. Firstly, our initial phase is highly symmetrical. Parameter $B < 0$, $C < 0$ (Fig. 1). In this case, there are no real solutions for the order parameter (20).
2. Parameters B and C will increase from ξ until $C = 0$ (Fig. 2) – a second-order phase transition occurs from a highly symmetric phase to a modulated phase (so-called MS1). The order parameter $\varphi(x)$ will have the form of an elliptic cosine of Jacobi (21). From (25), we can find the temperature of the phase transition. It will be equal to

$$q = -\frac{\gamma\sigma\xi_0}{2} - \frac{\sigma^2\xi_0{}^2}{4} \quad (26)$$

and will remain unchanged.

3. Further, $C > 0$ continues to grow (Fig. 3). The parameter k^2 is between 0 and 1/2.
4. Until B becomes equal to 0 ($k^2 = 1/2$) (Fig. 4). Since the parameter depends only on the external parameter ξ (24), according to Landau's definition of phase transitions, a first-order phase transition will occur. The value of parameter C will be equal to

$$C = \frac{q\acute{a}_{\pm}{}^{2}\sigma}{(2\sigma - 1)}. \qquad (27)$$

5. Changing the sign of B leads to new features – two maximum and one minimum appear (Fig. 5). The parameter is $1/2 < \mathrm{k}^2 < 1$.

6. Parameter C begins to decrease until it reaches zero again. A phase transition occurs between the two modulated phases MS1 and MS2 (Fig. 6) at the point

$$\xi = -\xi_0 - \frac{2\gamma}{\sigma}. \qquad (28)$$

The order parameter $\varphi(x)$ has the form of a bell soliton

$$\varphi(x) = \frac{a}{ch(bx)}. \qquad (29)$$

7. In order for parameter C to continue decreasing, it is necessary to change the temperature q (Fig. 7). The order parameter will have the form of delta amplitude (3).

8. The model (20) will continue to decline until the maximums of this model become equal to 0 (Fig. 8).

The equation between the parameters, in this case, will be

$$A = -\frac{B^2}{4C}. \qquad (30)$$

That is, the phase transition temperature from (28) will be found as (16).

Looking more closely at the evolution of the phase, we can conclude that there are restrictions on the values of ξ and σ.

Since it turns out that parameter C is not a monotonic function, it follows that $\sigma \in [0,\ 1/2]$ and only in this interval will the phase evolution take place. Since after (see Fig. 2) parameter C continues to grow, there is a restriction on the initial value ξ_0:

$$|\xi_0| > \frac{2\gamma}{\sigma}. \qquad (31)$$

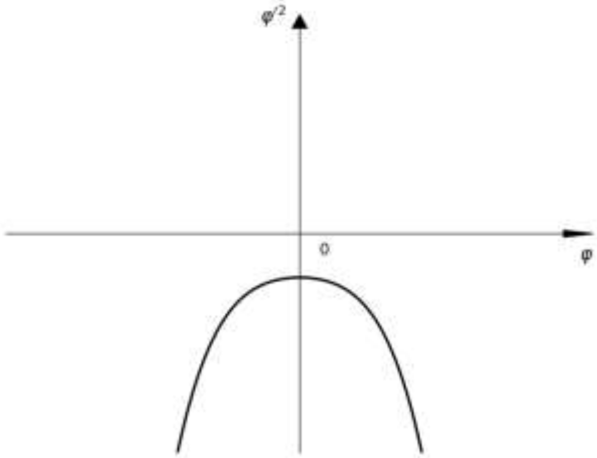

Fig. 1. Initial high-symmetric phase

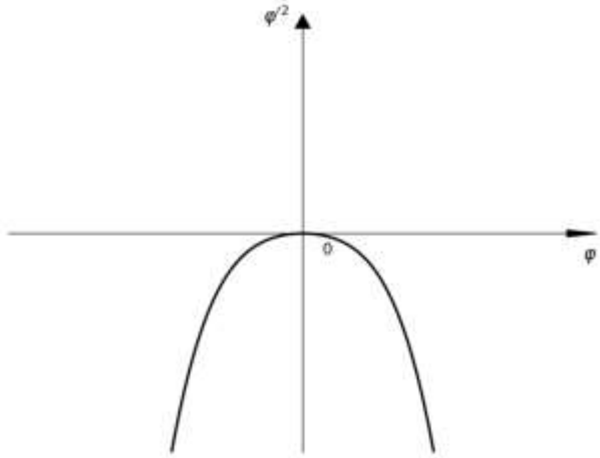

Fig. 2. Transition from the high-symmetric phase to the modulated phase (MS1)

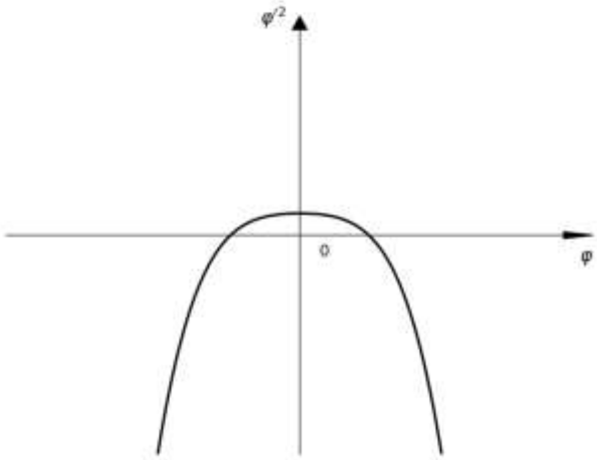

Fig. 3. Development of the MS1 phase ($0 < k^2 < 1/2$)

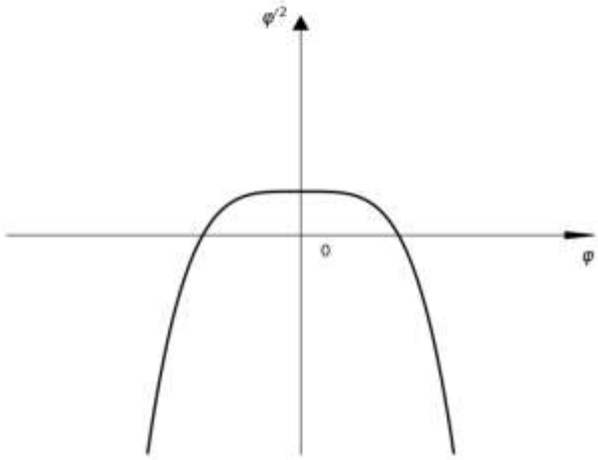

Fig. 4. Development of the MS1 phase ($k^2 = 1/2$)

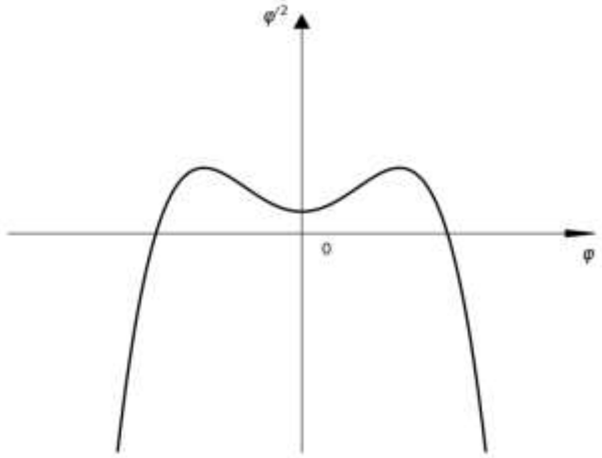

Fig. 5. Development of the MS1 phase ($1/2 < k^2 < 1$)

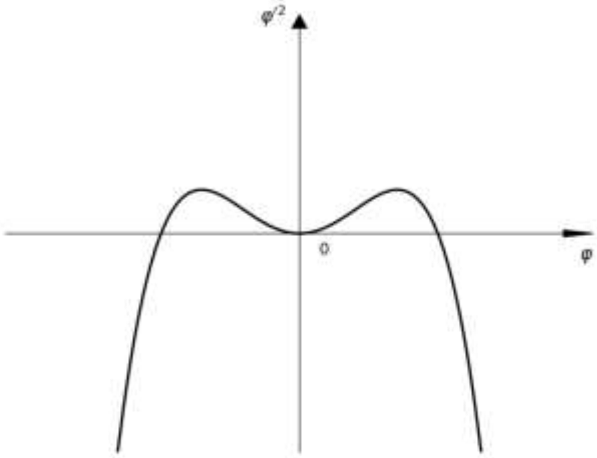

Fig. 6. Phase transition from phase MS1 to MS2

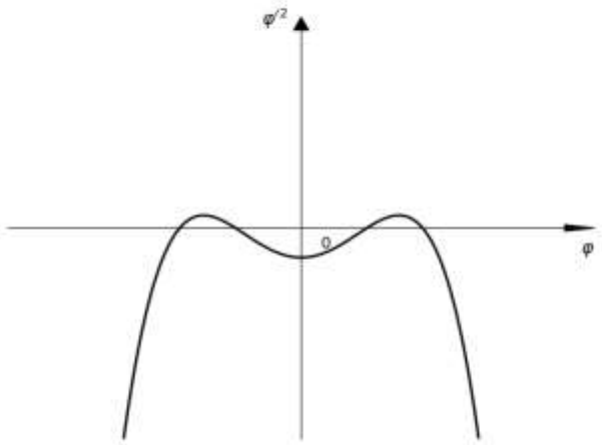

Fig. 7. Development of the MS2

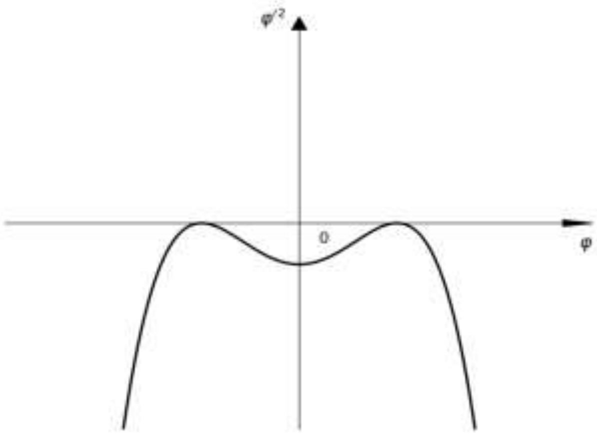

Fig. 8. Transition of phase MS2 to the final low-symmetric phase

CONCLUSIONS

Exact and partial exact solutions of a nonlinear fourth-order differential equation describing equilibrium states in systems with higher-order thermodynamic potential have been constructed.

It is shown that the order parameter can be represented by Jacobi elliptic functions: elliptic cosine and delta amplitude, and in certain limiting cases – by soliton solutions.

The interval of critical values of temperature parameters q_0 and q_1, at which a complete cycle of phase transitions occurs, has been found.

It has been established that the nature of phase evolution is determined by the ratio between the parameters ξ and σ, and only in the interval $\sigma \in [0, 1/2]$ is a sequential scenario of modulated transitions realised.

The results obtained provide a better understanding of the mechanisms of modulated phase formation and can be applied to describe structural transitions in pure materials and complex condensed systems.

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АНАЛІТИЧНІ РОЗВ'ЯЗКИ ТЕРМОДИНАМІЧНОГО ПОТЕНЦІАЛУ МОДЕЛІ МІХЕЛЬСОНА В РІВНОВАЖНИХ СИСТЕМАХ

А.Р. Шимановський, В.Ф. Клепіков

Досліджено точні розв’язки нелінійного диференційного рівняння четвертого порядку, яке випливає з варіаційного принципу для термодинамічного потенціалу з високими похідними. Показано, що окремі класи рішень можуть бути представлені через еліптичні функції Якобі типу дельта-амплітуди та еліптичного косинусу, а також у граничних випадках – через локалізовані белл-солітони. Проведено аналіз умов існування дійсних рішень, побудовано співвідношення між параметрами рівняння та фазовими переходами у модульовані стани. Зокрема, встановлено критичні температури переходів другого роду, що відповідають виникненню модульованих фаз (MS1, MS2), та наведено обмеження для параметрів термодинамічного потенціалу. Отримані результати узгоджуються з попередніми роботами та поглиблюють розуміння рівноважних станів у системах із сильною нелінійністю.