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ANALYSIS OF QED PROCESSES BY THE EXAMPLE OF THE PROCESS OF ELECTRON SCATTERING BY A MUON

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It is shown that the probabilities of second-order QED processes can be expressed through first-order process probabilities, the theory of which is extended to the case of virtual intermediate states. The derivations are carried out using the process of electron scattering by a muon as an example. The contribution of the polarized intermediate states to the probability of the process is investigated. It is shown that the polarized intermediate states make a fundamental contribution to the probability of the process.

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INTRODUCTION

The standard theory of QED processes [1] is formulated in such a way that the probability of a process of higher order in the fine-structure constant cannot be expressed in terms of the probabilities of the subprocesses from the sequence of which the process under study might be assembled. Should such a problem nevertheless be posed, it would be reasonable to expect that the probabilities of second-order processes can be written through the probabilities of first-order subprocesses in which one of the lines corresponds to a virtual state. As is well known, in the absence of an external field, first-order processes are forbidden by the 4-momentum conservation laws. However, if one of the particles is virtual (the particle lies off the mass shell), this prohibition is removed. We also note that, for a photonic intermediate state, besides the condition $\kappa^2 \neq 0$ it should be kept in mind that the polarization 4-vector e has three components [1] and that, in addition to the condition $e^2 = -1$, it satisfies the condition of transversality of the polarization with respect to the photon current: $(\kappa e)=0$.

Our choice of the process of electron scattering by a muon for the present analysis is motivated by the fact that it is described by a single Feynman diagram, so that the difficulties caused by the interference of amplitudes can be avoided. The process under study may be represented as a sequence of two subprocesses:

(a) emission of a virtual photon by the initial electron;

(b) absorption of the virtual photon by the initial muon.

The chronologically reversed sequence of the subprocesses is also possible and corresponds to the substitution $\kappa \to -\kappa$. However this does not affect the final results. Since the 4-momentum of the intermediate state enters the formulas quadratically as κ^2.

In order to compare the results obtained below, we write out here the expressions that describe the probability of the process of electron scattering by a muon (Fig. 1).

In particular, the 4-momentum conservation laws at the vertices have the form

$$\begin{cases} p_i = p_f + \kappa; \\ \Pi_i + \kappa = \Pi_f, \end{cases} \quad (1)$$

where $p_{\mathrm{i,f}} = \left(E_{\mathrm{i,f}}, \mathrm{p}_{\mathrm{i,f}}\right)$, $\Pi_{\mathrm{i,f}} = \left(\Xi_{\mathrm{i,f}}, \Pi_{\mathrm{i,f}}\right)$ are the 4-momenta of the initial and the final electron and muon; κ is the 4-momentum corresponding to the intermediate state ($\kappa^2 \neq 0$):

$$\kappa = p_i - p_f = \Pi_f - \Pi_i. \quad (2)$$

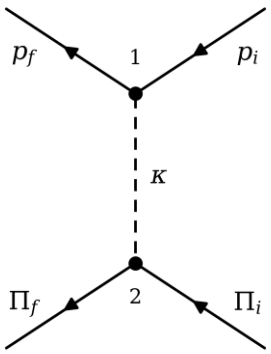


Fig. 1. Feynman diagram of the process of electron scattering by a muon

The amplitude of the scattering of an electron by a muon accompanied by the exchange of a virtual photon is determined by the expression

$$S_{fi} = \frac{\mathrm{i\pi e}^2}{V^2\sqrt{E_f E_i \Xi_f \Xi_i}} \frac{1}{\kappa^2} \left(B_\mu A^\nu\right) \times$$
$$\times (2\pi)^4 \delta^{(4)}\left(p_i + \mathrm{p}_f - \Pi_i - \Pi_f\right), \quad (3)$$

where V – the normalization volume; A_μ and B_μ are the electron and the muon currents:

$$A_\mu = \mathrm{u}_{p_f} \gamma_\mu u_{p_i}, B_\mu = \mathrm{u}_{\Pi_f} \gamma_\mu u_{\Pi_i}. \quad (4)$$

Here u_p and u_Π are the bispinors of the free electron and muon [1].

The argument of the δ-function in (3) determines the 4-momentum conservation law:

$$p_i + \Pi_i = p_f + \Pi_f. \quad (5)$$

The differential probability per unit time is

$$d\mathrm{w} = \frac{e^4}{4VE_i\Xi_i} \frac{1}{t^2} H d\Gamma, \quad (6)$$

where H is the relative probability of the process:

$$H = |(BA)|^2. \quad (7)$$

A direct calculation carried out in the standard way (see [1]), with averaging over the polarizations of the initial electron and muon and summation over the polarizations of the final electron and muon, yields

$$H"=2\left(2\left(s-m^2-m_\mu^2\right)^2+2ts+t^2\right), \quad (8)$$

where m, m_μ are the electron and muon masses; s, t, υ are the Mandelstam variables of the process under study [1]:

$$s=(p_i+\Pi_i)^2,\ t=\left(p_i-p_f\right)^2=\kappa^2,\ \upsilon=\left(p_i-\Pi_f\right)^2. \quad (9)$$

In expression (8) the variable υ has been eliminated by means of the relation

$$s+t+\upsilon=2m^2+2m_\mu^2. \quad (10)$$

There are also constraints on the admissible values of the parameters s and t:

$$s\geq s_{\min},\ s_{\min}=\left(m+m_{\,\mu}\right)^2,\ t_{\min}\leq t\leq 0, \quad (11)$$

$$t_{\min}=-\frac{\left(s-(m_\mu+\mathrm{m})^2\right)\left(s-(m_\mu-m)^2\right)}{s}. \quad (12)$$

The invariant differential $d\Gamma$ is determined by the element of the phase space into which the emission proceeds:

$$d\Gamma=\delta^{(4)}\left(p_i+p_{\,f}-\Pi_i-\Pi_f\right)\frac{d^3p_f d^3\Pi_f}{E_f\Xi_f}\rightarrow$$
$$\rightarrow\frac{2\pi dt}{\sqrt{\left(s-(m_\mu+\mathrm{m})^2\right)\left(s-(m_\mu-m)^2\right)}}, \quad (13)$$

where in the last expression the presence of the four-dimensional δ-function has made it possible to integrate over the momentum of the final muon and over the energy of the final electron, while the azimuthal symmetry in the centre-of-mass frame has made it possible to integrate over the azimuthal angle of the final electron.

Dividing (6) by the flux density of the incident particles

$$j=\frac{I}{VE_{\,i}\Xi_i},\ I=\sqrt{(p_i\Pi_i)^2-m^2m_\mu^2}=$$
$$=\frac{1}{2}\sqrt{\left(s-\left(m+m_{\,\mu}\right)^2\right)\left(s-\left(m-m_\mu\right)^2\right)} \quad (14)$$

and taking (13) into account, we arrive at the well-known expression for the differential cross-section of the process of electron scattering by a muon:

$$\frac{d\sigma}{dt}=\frac{2\pi e^4}{t^2\left(s-(m_\mu+m)^2\right)\left(s-(m_\mu-m)^2\right)}H. \quad (15)$$

1. PROBABILITY OF THE EMISSION OF A VIRTUAL PHOTON BY AN ELECTRON

Here we develop the theory of the process of emission of a virtual photon by means of the standard methods of QED, allowing for the fact that $\kappa^2\neq 0$ and that the virtual photon possesses three polarizations.

The amplitude of the emission of a virtual photon by an electron is

$$S_{fi}=-ie\sqrt{\frac{4\pi}{2\kappa_0 2E_f V 2E_i V}}\left(e_f A\right)\times$$
$$\times(2\pi)^4\delta^{(4)}\left(p_i-p_f-\kappa\right), \quad (16)$$

where u_p is the bispinor of the free electron; $p_{i,f}=\left(E_{i,f},p_{i,f}\right)$ are the 4-momenta of the initial (i) and the final (f) electrons; A_μ is the electron current (4).

The squared amplitude:

$$\left|S_{fi}\right|^2=e^{\,2}\frac{4\pi}{2\kappa_0 V 2E_f V 2E_i V}\mathrm{w}\frac{\mathrm{VT}}{(2\pi)^4}(2\pi)^8\times$$
$$\times\delta^{(4)}\left(p_i-p_f-\kappa\right), \quad (17)$$

where w is the relative probability of the process:

$$\mathrm{w}=|(eA)|^2=2\tfrac{1}{4}\mathrm{Sp}\left\{\left[p_f+m\right]e\left[p_i+m\right]e^*\right\}=$$
$$=2\left(2|(p_ie)|^2+(ee^*)\tfrac{\kappa^2}{2}\right). \quad (18)$$

In order to describe the polarization properties of the intermediate photon, we introduce a basis in the four-dimensional space:

$$\epsilon_0\equiv n_\kappa,\epsilon_{1,2}=e_{\,x,y}-\frac{(e_0e_{x,y})}{(ne_{\,0})}n,\ \epsilon_3=e_{\,0}-\frac{\lambda}{(ne_{\,0})}n, \quad (19)$$

where e_0,e_x,e_y,e_z are the ordinary basis of Minkowski space; $n_\kappa=\kappa/\sqrt{|\kappa^2|}$, $n=e_{\,0}+e_{\,z}$, $n^2=0$; the 4-vector e_3^μ is related to the remaining 4-vectors of the basis by the relation

$$\epsilon_3^\mu=e^{\,\mu\nu\rho\tau}\epsilon_{0,\nu}\epsilon_{1,\rho}\epsilon_{2,\tau}, \quad (20)$$

where $e^{\mu\nu\rho\tau}$ is the four-dimensional Levi-Civita symbol. The vectors ϵ_0, ϵ_1, ϵ_2, ϵ_3 satisfy the standard conditions imposed on the unit vectors of a basis in the four-dimensional space:

$$\epsilon_0^2=-\epsilon_3^2=\lambda;\ \lambda=\pm 1;\ \epsilon_1^2=\epsilon_2^2=-1. \quad (21)$$

Note that the choice of the polarization vectors in the form (19) is not unique. We could, for instance, construct the basis from the independent 4-vectors of the problem, for example from the 4-momenta of the initial electron and muon, p_i and Π_i, and from the 4-momentum κ of the intermediate photon. Three orthonormal basis vectors can be built from them, the fourth one being obtained by means of formula (20). The choice of the basis in the form (19) is dictated by the interest of the authors in constructing an analogous theory for second-order processes in the field of a wave [2]. For that problem the choice of the basis (19) is the most natural one, since a direction corresponding to the wave vector of the field is singled out in the four-dimensional space. The absence of such a preferred direction in the process considered here manifests itself in the absence, from expression (8) for the relative probability, of any variables associated with that direction.

Let us write the product $\left(e_fA\right)$ in the new basis:

$$\left(e_fA\right)=-\left(e_f\epsilon_1\right)(\mathrm{A}\epsilon_1)-$$
$$-(\mathrm{e}\epsilon_2)(\mathrm{A}\epsilon_2)-\lambda(\mathrm{e}\epsilon_3)(\mathrm{A}\epsilon_3), \quad (22)$$

where the orthogonality of the photon and the electron currents has been taken into account: $(e_0A)=(\kappa\mathrm{A})/\sqrt{|\kappa^2|}=0$.

The squared modulus of expression (22) may be represented in the form

$$\mathrm{w}=\left|\left(e_fA\right)\right|^2=\tfrac{\mathrm{w}_1+\mathrm{w}_2+\lambda\mathrm{w}_3}{3}+$$
$$+\sum_{i=1}^{2}\sum_{j=i+1}^{3}\left(\frac{1}{3}\xi_1^{(\mathrm{ij})}\left(\mathrm{w}_i-\lambda_j\mathrm{w}_j\right)+\right.$$
$$\left.+\frac{\lambda_j}{2}\left(\xi_2^{(ij)}\left(\mathrm{w}_{+\frac{\pi}{4}}^{(ij)}-\mathrm{w}_{-\frac{\pi}{4}}^{(ij)}\right)+\xi_3^{(ij)}\left(\mathrm{w}_-^{(ij)}-\mathrm{w}_+^{(ij)}\right)\right)\right), \quad (23)$$

where $\lambda_1=\lambda_2=1$; $\lambda_3=\lambda=-1$; and $\xi_1^{(ij)}$, $\xi_2^{(ij)}$, $\xi_3^{(ij)}$ are the generalization of the Stokes parameters to the case of three polarizations [2]:

$$\xi_1^{(\mathrm{ij})}=\left|\left(e_f\epsilon_1\right)\right|^2-\lambda_j\left|\left(e_f\epsilon_2\right)\right|^2, \quad (24)$$

$$\xi_2^{(\mathrm{ij})}=2\Re\left(\left(e_f\epsilon_i\right)\left(e_f\epsilon_j\right)\right), \quad (25)$$

$$\xi_3^{(ij)} = 2\Im\left((e_f\epsilon_i)(e_f\epsilon_j)\right). \quad (26)$$

The first term in (23) corresponds to an unpolarized intermediate state:

$$\mathrm{w}_{\mathrm{unres}} = \mathrm{w}_1 + \mathrm{w}_2 + \lambda\mathrm{w}_3 = \\ = -2\frac{1}{4}\mathrm{Sp}\{[p_f+m]\gamma_\mu[p_i+m]\gamma^\mu\}. \quad (27)$$

The calculations give

$$\mathrm{w}_{\mathrm{unres}} = 4\left(-\frac{\kappa^2}{2} - m^2\right). \quad (28)$$

Within the range of variation of the parameter t specified by (11) this expression may take negative values. Hence, having extended the theory of the processes to virtual states, we are faced with the fact that the probabilities corresponding to the polarization effects may become negative. It is therefore more appropriate to call them quasi-probabilities, and we note that in quantum mechanics this is by no means an unusual situation (see, for example, Ref. [3]).

In expression (23) the terms proportional to $\mathrm{w}_1 - \mathrm{w}_2$, $\mathrm{w}_1 - \mathrm{w}_3$, $\mathrm{w}_2 - \mathrm{w}_3$ correspond to the polarization effects associated with the linear polarizations along the polarization vectors e_1, e_2, e_3. The probabilities corresponding to these polarizations, evaluated from formulas (18), have the form

$$\mathrm{w}_j = 2\left(2(p_i\epsilon_j)^2 - \lambda_j\frac{\kappa^2}{2}\right), j = 1,2,3. \quad (29)$$

Let us introduce the 4-vector

$$g = \frac{p_f}{(np_f)} - \frac{p_i}{(np_i)}, \quad (30)$$

$$-g^2 = \frac{16m^2}{(n\kappa)^2}u_{\max}^2\frac{u}{u_{\max}}\left(1 - \frac{u}{u_{\max}}\right), \quad (31)$$

where u and u_{max} are invariant parameters:

$$u = \frac{(n\kappa)^2}{4(np_i)(np_f)},\ 0 \le u \le u_{\max}, u_{\max} = -\frac{t}{2m^2}. \quad (32)$$

We take into account that $(g\epsilon_1) = -|g|\sin\theta\cos\psi = -\sqrt{-g^2}\cos\psi$, $(g\epsilon_2) = -\sqrt{-g^2}\sin\psi$, where θ and ψ are the polar and the azimuthal angles of the vector g in the spherical frame of reference whose axis is directed along e_3. On the other hand

$$(g\epsilon_{1,2}) = (p_i\epsilon_{1,2})\frac{(n\kappa)}{(np_f)(np_i)}. \quad (33)$$

Then find

$$(p_i\epsilon_1) = -m\sqrt{\frac{u_{\max}}{u} - 1}\cdot\cos\psi, \quad (34)$$

$$(p_i\epsilon_2) = -m\sqrt{\frac{u_{\max}}{u} - 1}\cdot\sin\psi. \quad (35)$$

Also

$$(p_i\epsilon_3) = \pm\frac{1}{2}\sqrt{(-t)\frac{1+\mathrm{u}}{u}}. \quad (36)$$

Note that a single value of the variable u corresponds to two values of $(p_i\epsilon_3)$.

Making use of the expressions obtained for the terms that describe the linear polarization effects, we find

$$\mathrm{w}_1 - \mathrm{w}_2 = 4((p_i\epsilon_1)^2 - (p_i\epsilon_2)^2) = \\ = 4m^2\left(\frac{u_{\max}}{u} - 1\right)\cos2\psi, \quad (37)$$

$$\mathrm{w}_1 - \lambda\mathrm{w}_3 = 4((p_i\epsilon_1)^2 - \lambda(p_i\epsilon_3)^2) = \\ = 4m^2\left(\left(\frac{u_{\max}}{u} - 1\right)\cos\psi + \frac{1+\mathrm{u}}{u}u_{\max}\right), \quad (38)$$

$$\mathrm{w}_2 - \lambda\mathrm{w}_3 = 4((p_i\epsilon_2)^2 - \lambda(p_i\epsilon_3)^2) = \\ = 4m^2\left(\left(\frac{u_{\max}}{u} - 1\right)\sin\psi + \frac{1+\mathrm{u}}{u}u_{\max}\right). \quad (39)$$

For the relative probabilities corresponding to the polarization vectors at the angles $\pm\pi/4$ in the planes $\epsilon_1\epsilon_2$, $\epsilon_1\epsilon_3$, $\epsilon_2\epsilon_3$:

$$e_{\pm\pi/4}^{(\mathrm{ij})} = \frac{1}{\sqrt{2}}(\epsilon_i \pm \lambda_j\epsilon_j), \quad (40)$$

we obtain

$$\mathrm{w}_{\pm\pi/4}^{(12)} = \left|\left(e_{\pm\frac{\pi}{4}}^{(12)}A\right)\right|^2 = 2\left(((p_i\epsilon_1) \pm (p_i\epsilon_2))^2 - \frac{\kappa^2}{2}\right), \quad (41)$$

$$\mathrm{w}_{\pm\pi/4}^{(\mathrm{j3})} = \left|\left(e_{\pm\pi/4}^{(\mathrm{j3})}A\right)\right|^2 = \\ = 2\left(\left((p_i\epsilon_j) \pm (p_i\epsilon_3)\right)^2 - \frac{\kappa^2}{2}\delta_{1\lambda}\right),\ j = 1,2. \quad (42)$$

For the effects associated with the polarization at the angles $\pm\pi/4$ in the planes e_1e_2, e_1e_3, e_2e_3 we obtain

$$\mathrm{w}_{\pi/4}^{(12)} - \mathrm{w}_{-\pi/4}^{(12)} = 8(p_i\epsilon_1)(p_i\epsilon_2) = \\ = 4m^2\left(\frac{u_{\max}}{u} - 1\right)\sin2\psi, \quad (43)$$

$$\mathrm{w}_{\pi/4}^{(13)} - \mathrm{w}_{-\pi/4}^{(13)} = 8(p_i\epsilon_1)(p_i\epsilon_3) = \\ = -4m^2\frac{u_{\max}}{u}\sqrt{(1+\mathrm{u})\left(1 - \frac{u}{u_{\max}}\right)}\cos\psi, \quad (44)$$

$$\mathrm{w}_{\pi/4}^{(23)} - \mathrm{w}_{-\pi/4}^{(23)} = 8(p_i\epsilon_2)(p_i\epsilon_3) = \\ = -4m^2\frac{u_{\max}}{u}\sqrt{(1+\mathrm{u})\left(1 - \frac{u}{u_{\max}}\right)}\sin\psi. \quad (45)$$

In a similar way, the relative probabilities corresponding to the circular polarization vectors in the planes $\epsilon_1\epsilon_2$, $\epsilon_1\epsilon_3$, $\epsilon_2\epsilon_3$,

$$e_{\pm}^{(\mathrm{ij})} = \frac{1}{\sqrt{2}}(\epsilon_i \pm \lambda i\epsilon_{\ 3}) \quad (46)$$

have the form

$$\mathrm{w}_{\pm}^{(12)} = \left|\left(e_{\pm}^{(12)}A\right)\right|^2 = 2\left((p_i\epsilon_1)^2 + (p_i\epsilon_2)^2) - \frac{\kappa^2}{2}\right), \quad (47)$$

$$w_{\pm}^{(j3)} = \left|\left(e_{\pm}^{(j3)}A\right)\right|^2 = \\ = 2\left(\left((p_i\epsilon_j)^2 + (p_i\epsilon_3)^2\right) - \frac{\kappa^2}{2}\delta_{1\lambda}\right),\ j = 1,2. \quad (48)$$

It follows from this that none of the effects associated with the circular polarization shows up:

$$\mathrm{w}_{-}^{(\mathrm{ij})} - \mathrm{w}_{+}^{(ij)} = 0, i = 1,2;\ j = i+1,..,3. \quad (49)$$

2. PROBABILITIES OF THE ABSORPTION OF THE INTERMEDIATE PHOTON BY A MUON

For the second subprocess (the absorption of the intermediate photon by the muon) analogous expressions may be obtained by the simple substitution $p_i \to \Pi_i$, $p_f \to \Pi_f$ (and also $\kappa \to -\kappa$). By analogy with (32), we introduce the corresponding invariant parameters of the second subprocess, the absorption of the intermediate photon by the muon:

$$U = \frac{(\mathrm{n}\kappa)^2}{4(\mathrm{n}\Pi_i)(\mathrm{n}\Pi_f)}, 0 \le U \le U_{\max}, U_{\max} = -\frac{t}{(2m_\mu^2)}. \quad (50)$$

The projections of the muon 4-momentum onto the unit vectors of the basis (20) have the form

$$(\Pi_i e_0) = \lambda m_{\ \mu}\sqrt{-\lambda\mathrm{U}_{\max}}, \quad (51)$$

$$(\Pi_i e_1) = -m_\mu\sqrt{\frac{U_{\max}}{U} - 1}\cos\psi_2, \quad (52)$$

$$(\Pi_i e_2) = -m_\mu\sqrt{\frac{U_{\max}}{U} - 1}\sin\psi_2, \quad (53)$$

$$(\Pi_i e_3) = \pm m_\mu\sqrt{(-\lambda)U_{\max}\left(1 + \frac{1}{U}\right)}, \quad (54)$$

where ψ_2 is the azimuthal angle of the vector $g(2) = \Pi_f/(k\Pi_f) - \Pi_i/(k\Pi_i)$.

Let us write out the formulas for the polarization effects associated with the photonic intermediate state that refer to the second subprocess. In particular, for an unpolarized intermediate state,

$$\mathrm{w}_{\mathrm{unpol}}(2) = 4\left(\frac{-t^2}{2} - m_\mu^2\right), \qquad (55)$$

where $2 \equiv (\Pi_f, \Pi_i)$ indicates that the quantity in question belongs to the second subprocess. In what follows, the fact that a quantity belongs to the first subprocess will be indicated in the same manner: $1 \equiv (p_f, p_i)$.

The relative probabilities that describe the effects associated with the linear polarization of the intermediate state are

$$\mathrm{w}_1(2) - \mathrm{w}_2(2) = 4((\Pi_i e_1)^2 - (\Pi_i e_2)^2) = \\ = 4m_\mu^2\left(\frac{U_{\max}}{U} - 1\right)\cos 2\psi_2, \qquad (56)$$

$$\mathrm{w}_1(2) - \lambda \mathrm{w}_3(2) = 4((\Pi_i e_1)^2 - \lambda(\Pi_i e_3)^2) = \\ = 4m_\mu^2\left(\left(\frac{U_{\max}}{U} - 1\right)\cos\psi_2 + \frac{1+\mathrm{U}}{U}U_{\max}\right), \qquad (57)$$

$$\mathrm{w}_2(2) - \lambda \mathrm{w}_3(2) = 4((\Pi_i e_2)^2 - \lambda(\Pi_i e_3)^2) = \\ = 4m_\mu^2\left(\left(\frac{U_{\max}}{U} - 1\right)\sin\psi_2 + \frac{1+\mathrm{U}}{U}U_{\max}\right). \qquad (58)$$

The relative probabilities that describe the effects associated with the polarization of the intermediate state at the angles $\pm\pi/4$ are

$$\mathrm{w}_{\pi/4}^{(12)}(2) - \mathrm{w}_{-\pi/4}^{(12)}(2) = 4(\Pi_i e_1)(\Pi_i e_2) = \\ = 2m_\mu^2\left(\frac{U_{\max}}{U} - 1\right)\sin 2\psi_2, \qquad (59)$$

$$\mathrm{w}_{\pi/4}^{(13)}(2) - \mathrm{w}_{-\pi/4}^{(13)}(2) = 4(\Pi_i e_1)(\Pi_i e_3) = \\ = -4m_\mu^2\sqrt{(1+U)\left(\frac{U_{\max}}{U} - 1\right)}\cos\psi_2, \qquad (60)$$

$$\mathrm{w}_{\pi/4}^{(23)}(2) - \mathrm{w}_{-\pi/4}^{(23)}(2) = 4(\Pi_i e_2)(\Pi_i e_3) = \\ = -4m_\mu^2\sqrt{(1+U)\left(\frac{U_{\max}}{U} - 1\right)}\sin\psi_2, \qquad (61)$$

$$\mathrm{w}_-^{(\mathrm{ij})} - \mathrm{w}_+^{(\mathrm{ij})} = 0,\ i = 1,2;\ j = i+1, \ldots, 3\,. \qquad (62)$$

3. CONTRIBUTION OF THE POLARIZATION STATES TO THE PROBABILITY OF ELECTRON SCATTERING BY A MUON

In order to express the probability of the process of electron scattering by a muon through the probabilities of the subprocesses, we rewrite the relative probability H (7) in the basis (19) and take into account the orthogonality of the electron and the muon currents to the photon current ($(A_0 = \mathrm{B}_0 = 0$):

$$\mathrm{H} = |B_1|^2|A_1|^2 + |B_2|^2|A_2|^2 + |B_3|^2|A_2|^2 + \\ + 2\Re(B_2B_1A_2A_1 + \\ + \lambda(B_3B_1A_3A_1 + \mathrm{B}_3B_2A_3A_2)). \qquad (63)$$

By means of algebraic transformations this expression can also be reduced to the form

$$H = \frac{1}{3}(|B_1|^2 + |B_2|^2 + \lambda|B_3|^2) \times \\ \times (|A_1|^2 + |A_2|^2 + \lambda|A_3|^2) + \\ + \sum_{i=1}^{2}\sum_{j=i+1}^{3}\left(\frac{1}{3}\left(|A_i|^2 - |A_j|^2\right)\left(|B_i|^2 - |B_j|^2\right) + \right. \\ + \frac{\lambda_j}{2}\left(\left((B_iB_j) + (B_jB_i)\right)\left((A_iA_j) + (A_jA_i)\right) + \right. \\ \left.\left. + \left((B_iB_j) - (B_jB_i)\right)\left((A_iA_j) - (A_jA_i)\right)\right)\right). \qquad (64)$$

Proceeding from this expression, we are able to express H in terms of the relative probabilities that describe the polarization properties of the intermediate states of the subprocesses:

$$H = \frac{1}{3}\mathrm{w}_{\mathrm{unpolar}}(2)\mathrm{w}_{\mathrm{unpolar}}(1) + \\ + \sum_{i=1}^{2}\sum_{j=i+1}^{3}\left(\frac{1}{3}\left(\mathrm{w}_i(2) - \lambda_j\mathrm{w}_j(2)\right)\left(\mathrm{w}_i(1) - \lambda_j\mathrm{w}_j(1)\right)\right. \\ + \frac{\lambda_j}{2}\left(\left(\mathrm{w}_{\frac{\pi}{4}}^{(ij)}(2) - \mathrm{w}_{-\frac{\pi}{4}}^{(ij)}(2)\right)\left(\mathrm{w}_{\frac{\pi}{4}}^{(ij)}(1) - \mathrm{w}_{-\frac{\pi}{4}}^{(ij)}(1)\right)\right. \\ \left.\left.\left. + \left(\mathrm{w}_+^{(ij)}(2) - \mathrm{w}_-^{(ij)}(2)\right)\left(\mathrm{w}_+^{(ij)}(1) - \mathrm{w}_-^{(ij)}(1)\right)\right)\right), \qquad (65)$$

where the relative probabilities describing the polarization effects that refer to the first subprocess are determined by formulas (28), (37)–(39), (43)–(45), and (48), while the relative probabilities describing the polarization effects that refer to the second subprocess are determined by formulas (56)–(62).

Thus, the relative probability of the second-order process of the scattering of an electron by a muon and, together with it, the differential probability of the process, has been written through the relative probabilities of the first-order subprocesses that allow for the polarization effects corresponding to the photonic intermediate state.

The first term in (65) corresponds to the contribution of the unpolarized intermediate state:

$$H_{\mathrm{unpolar}} = \frac{1}{3}\mathrm{w}_{\mathrm{unpolar}}(2)\mathrm{w}_{\mathrm{unpolar}}(1) = \\ = \frac{16}{3}\left(\frac{-t}{2} - m^2\right)\left(\frac{-t}{2} - m_\mu^2\right). \qquad (66)$$

The relative probability corresponding to the effects associated with the linear polarization of the intermediate state is

$$H_{\mathrm{linear}} = \frac{1}{3}\sum_{i=1}^{2}\sum_{j=i+1}^{3}\left(\mathrm{w}_i(2) - \lambda_j\mathrm{w}_j(2)\right) \times \\ \times \left(\mathrm{w}_i(1) - \lambda_j\mathrm{w}_j(1)\right). \qquad (67)$$

The relative probability corresponding to the effects associated with the polarization of the intermediate state at the angle $\pm\pi/4$ is

$$H_{\pi/4} = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=i+1}^{3}\lambda_j\left(\mathrm{w}_{\frac{\pi}{4}}^{(ij)}(2) - \mathrm{w}_{-\frac{\pi}{4}}^{(ij)}(2)\right) \times \\ \times \left(\mathrm{w}_{\frac{\pi}{4}}^{(ij)}(1) - \mathrm{w}_{-\frac{\pi}{4}}^{(ij)}(1)\right). \qquad (68)$$

The effects associated with the circular polarization of the intermediate state do not show up:

$$H_{\mathrm{circ}} = \frac{1}{2}\sum_{\mathrm{i}=1}^{2}\sum_{\mathrm{j}=\mathrm{i}+1}^{3}\lambda_j\left(\mathrm{w}_+^{(ij)}(2) - \mathrm{w}_-^{(ij)}(2)\right) \times \\ \times \left(\mathrm{w}_+^{(ij)}(1) - \mathrm{w}_-^{(ij)}(1)\right) = 0. \qquad (69)$$

Attention should also be paid to the contribution of the third polarization. Isolating in formula (65) the terms associated with it, we obtain

$$H_3 = \mathrm{W}_3(2)W_3(1) + \frac{\lambda}{2}\sum_{i=1}^{2}\left(W_{\frac{\pi}{4}}^{(i,3)}(2) - W_{-\frac{\pi}{4}}^{(i,3)}(2)\right) \times \\ \times \left(W_{\frac{\pi}{4}}^{(i,3)}(1) - W_{-\frac{\pi}{4}}^{(i,3)}(1)\right), \qquad (70)$$

where the terms associated with the circular polarization have been dropped because their contribution vanishes (see Eq. (69)).

On the other way, if the expressions describing the polarization effects associated with the intermediate states of the subprocesses are substituted into the relative probability (65), it can be shown that the latter coincides with the relative probability of electron scattering by a muon (8). In particular, the dependence on the variables u, ψ_1, U, ψ_2 thereby disappears, which is only natural, since the process under study has no preferred direction, and the variables associated with such a direction drop out of the total probability.

Let us note that the parameter s is related to the other parameters describing the subprocesses by the relation

$$s = m^2 + m_\mu^2 + 2\left(-\frac{t}{4} - (p_i e_1)(\Pi_i e_1) - \right.$$
$$\left. -(p_i e_2)(\Pi_i e_2) - \lambda(p_i e_3)(\Pi_i e_3)\right), \qquad (71)$$

where $(p_i e_j)$, $(\Pi_i e_j)$, $j = 1,2,3$, are determined by expressions (34)–(36) and (52)–(54), so that $s = s(t, u, \psi_1, U, \psi_2)$.

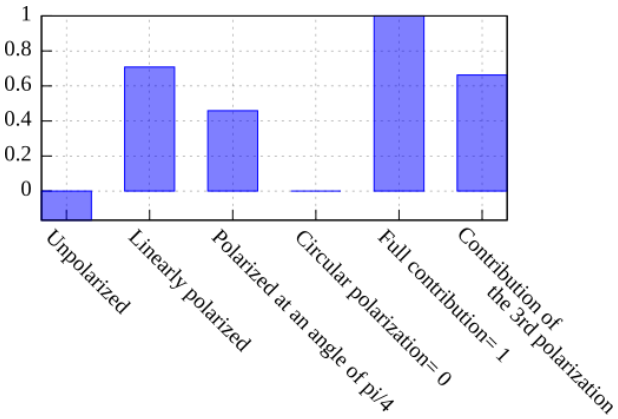


Fig. 2. Relative contribution of the polarized intermediate states to the probability of the process of electron scattering by a muon

We note that in the region of admissible values of t the contribution of the unpolarized intermediate state is negative, whence it may be stated that the process is realized solely owing to the contribution of the linearly polarized state and of the polarization at the angle $\pm\pi/4$, the contribution of the circular polarization being zero. This can likewise be verified by direct calculation. The contribution of the polarized states to the relative probability of the process is shown in Fig. 2. The calculations are presented for the values $t = -5$, $u = u_{max}/2$, $U = U_{max}/2$, $\psi_1 = \psi_2 = \pi/4$. A similar picture is observed for other values of the parameters.

Fig. 2 also presents the relative contribution of the third polarization (70). This contribution is substantial and, as may be seen, even exceeds the total contribution of the polarizations along $\epsilon_{1,2}$.

CONCLUSIONS

The results of the study allow the following conclusions to be drawn.

1. The probability of the process of electron scattering by a muon can be written through the probabilities of first-order subprocesses that allow for the polarization properties of the intermediate state.

2. The contribution of the polarized intermediate state to the total probability is fundamental, since the unpolarized photonic intermediate state makes a negative contribution to the probability.

3. The contribution of the third (additional) polarization is substantial and may exceed the total contribution of the other polarizations.

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АНАЛІЗ QED-ПРОЦЕСІВ НА ПРИКЛАДІ ПРОЦЕСУ РОЗСІЮВАННЯ ЕЛЕКТРОНА НА МЮОНІ

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На прикладі процесу розсіювання електрона на мюоні показано, що ймовірності QED-процесів 2-го порядку можна виразити через ймовірності процесів 1-го порядку, теорія яких розширена на випадок віртуальних проміжних станів. Досліджений вклад поляризованих проміжних станів у ймовірність процесу. Показано, що поляризовані проміжні стани вносять фундаментальний вклад у ймовірність процесу.