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PAULI-LUBANSKI PSEUDOVECTOR AND ORTHONORMAL TETRAD

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As is known, eigenvalues of helicity, defined as proportionality coefficient between Pauli-Lubanski pseudovector and momentum, are used to label massless unitary representations of the Poincare group. However, there is a problem to get manifestly Lorentz-invariant expression for it in any other way but in the twistor approach. In this work it is shown that application of orthonormal tetrad in Minkowski space-time allows obtaining such an expression in terms of the Pauli-Lubanski pseudovector. Use of orthonormal tetrad also allows expressing in the Lorentz-invariant way spin three-vector of massive particle via the Pauli-Lubanski pseudovector.

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INTRODUCTION

Positive-energy unitary irreducible representations of restricted Poincare group $ISO_0(1,3)$, which is the semi-direct product of the proper orthochronous Lorentz group $SO_0(1,3)$ and Abelian group of space-time translations, constitute group-theoretic basis of relativistic quantum mechanics and field theory in four-dimensional Minkowski space-time. Systematic study of such representations was initiated by E. Vigner [1] (for modern textbook treatment see, e.g. [2, 3]). Important role in their characterization is played by two Casimir operators of the *iso*(1,3) Poincare algebra

$$[M_{kl}, M_{mn}] = i(\eta_{km}M_{ln} - \eta_{kn}M_{lm} + \eta_{ln}M_{km} - \eta_{lm}M_{kn}),$$
$$[M_{kl}, P_m] = i(\eta_{km}P_l - \eta_{lm}P_k), \quad [P_m, P_n] = 0,$$

where η_{mn}=(-1,+1,+1,+1) is Minkowski metric. These are quadratic Casimir operator P^2 and quartic one W^2, where

$$W^m = \frac{1}{2}\varepsilon^{mkln}M_{kl}P_n, \ k,l,m,n=0,1,2,3, \ \varepsilon^{0123}=1 \quad (1)$$

is the Pauli-Lubanski pseudovector [4, 5]. From the definition (1) there follows its orthogonality to P_m and commutation relations

$$[P_m, W_n] = 0, \ [M_{kl}, W_m] = i(\eta_{km}W_l - \eta_{lm}W_k),$$
$$[W_m, W_n] = i\varepsilon_{mnkl}W^k P^l. \quad (2)$$

It is known explicit expression for W^2 in terms of generators of the *iso*(1,3) algebra

$$W^2 = M_{kl}P^l M^{kn}P_n - \frac{1}{2}M_{kl}M^{kl}P^2 \quad (3)$$

that generalizes *so*(1,3) Casimir operators $M^{mn}M_{mn}$ and $\varepsilon^{klmn}M_{kl}M_{mn}$.

In general, the number of independent Casimir operators of *iso*(1,*D*-1) algebra equals integer part of (*D*+1)/2 [6] and P^2 together with W^2 in the form (3) remain Casimir operators for arbitrary *D*>4. However, only for *D*=4 W^2 can be written as the Pauli-Lubanski pseudovector squared. Let us note that recently there revived an interest to study higher-dimensional generalizations of the Pauli-Lubanski pseudovector and higher-order Casimirs of Poincare algebras for *D*>4 (see, e.g. [7–9]). At the same time, for instance, for *iso*(1,4) algebra tensor generalization of the Pauli-Lubanski pseudovector and Casimir operators are known for a long time [10–12].

For physically important class of massless representations of $ISO_0(1,3)$ with zero eigenvalues of both Casimirs, their characterization is achieved with the aid of (half-)integer eigenvalues of helicity Λ. It is defined as the proportionality coefficient between the Pauli-Lubanski pseudovector and momentum

$$W^m = \Lambda p^m. \quad (4)$$

Helicity is the Poincare-invariant quantity but it is hardly possible to construct manifestly invariant expression for it just from generators of the Poincare algebra. Such an expression can be obtained in point-particle mechanics, e.g. in the spinning particle model with local world-line supersymmetry [13]. It is based on explicit realization of the spin degrees of freedom that allows expressing *so*(1,3) generators in terms of them. At the same time, manifestly conformally invariant expression for helicity naturally arises in twistor theory [14, 15]. Fundamental role in this theory is played by complex twistor space and space-time is an emergent object. The aim of this note is to show that introducing components of orthonormal tetrad [16, 17] allows to derive manifestly Lorentz-invariant expression for helicity also in the space-time approach, irrespective of the way used to describe spin degrees of freedom.

For massive representations that constitute another important class of unitary representations of Poincare group, introduction of the tetrad allows to get manifestly Lorentz-invariant general expression for spin three-vector. Its components satisfy *so*(3) algebra relations independently of the choice of spin variables.

In the next section it is briefly recapitulated the traditional approach to massive and massless representations of the Poincare group. Section 3 is devoted to overview of conformally invariant description of helicity in twistor theory. Then in Section 4 definition of the orthonormal tetrad and its realization in terms of spinor dyad is recalled as well as their group-theoretic interpretation as Lorentz harmonics. This Section is culminated by derivation of manifestly Lorentz-invariant expression for helicity and spin three-vector. Brief summary of obtained results is contained in concluding Section.

1. MASSIVE AND MASSLESS REPRESENTATIONS OF THE POINCARE GROUP

According to Wigner classification of unitary irreducible representations of the Poincare group, there exist five distinct classes of representations. Of the primary importance for physical applications are two of them. It is possible to label them by eigenvalues of the maximal set of mutually commuting generators and Casimir operators. Concentrating on realization of such representations in the infinite-dimensional Hilbert space of single-particle states, momentum components and Casimirs can be chosen to span such a set. For both classes of representations it is assumed to hold the energy positivity condition $p^0>0$. This condition can be cast in the form of additional topological label $\mathrm{sign}(p^0)=+1$ that selects either upper sheet of the hyperbolic surface $p^2+m^2=0$ in momentum space or the future-pointing half of the light cone $p^2=0$ for $m=0$.

The first class includes representations that correspond to massive particles with spin s and $p^2+m^2=0$, for which $W^2=m^2s(s+1)$. Derivation of eigenvalue of the Pauli-Lubanski pseudovector squared is commonly performed in the particle's rest frame, in which its momentum takes value $p_0^m=(m,0,0,0)$ in natural units $c=\hbar=1$. In this frame commutation relations (2) of the components of Pauli-Lubanski pseudovector $W_0=0$, $W_i=mS_i$, $S_i=\frac{1}{2}\varepsilon_{ijk}M_{jk}$ reduce to those of $so(3)=su(2)$ algebra

$$[S_i,S_j]=i\varepsilon_{ijk}S_k$$

that supports identification of the 3-vector S_i as spin operator of the massive particle. On the other hand, $so(3)$ is Lie algebra of the little group of stability of four-momentum in the rest frame and it plays crucial role in Wigner's method of induced representations of the Poincare group. In the rest frame Pauli-Lubanski pseudovector squared reduces to quadratic Casimir operator of the $so(3)$ algebra of space rotations, eigenvalue of which on one-particle states with definite spin s equals $s(s+1)$.

To the second class there belong representations with $p^2=0$ and $W^2=0$ describing massless particles. Apart from four-momentum these representations are characterized by eigenvalue of the helicity operator Λ that takes (half-)integer values $\lambda=0, \pm1/2, \pm1,\ldots$ Modulus of helicity $|\lambda|$ is known as spin of massless particle. For massless particles with definite momentum, three conditions $p^2=W^2=W^mp_m=0$ imply that $W^m=\Lambda p^m$. In the frame, in which particle's momentum is oriented along the first coordinate axis $p_0^m=(E,E,0,0)$, where E is its energy, from the definition (1) it follows that components of the Pauli-Lubanski pseudovector equal

$$W^0=W_1=ES_1,\ W_2=ET_1,\ W_3=ET_2.$$

Here $T_1=S_2-N_3$, $T_2=S_3-N_2$, and $N_i=M_{0i}$ are generators of the Lorentz boosts. In this frame commutation relations of the components of Pauli-Lubanski pseudovector

$$[T_1,T_2]=0,\ [T_1,S_1]=iT_2,\ [T_2,S_1]=-iT_1$$

correspond to the *iso(2)* algebra of rotations and translations of the two-dimensional plane. This *iso(2)* algebra is the Lie algebra of little group of stability of null momentum in the chosen frame. In this frame the Casimir operator W^2 reduces to $T_1^2+T_2^2$. For considered class of representations its eigenvalue is zero but there exists another class of massless representations with real positive eigenvalues. Such representations are called continuous-spin representations, which existence is possible due to commutativity of T_1 and T_2. They were first studied by Wigner [18] and recently became the subject of active exploration (see, e.g. [19, 20]), though have no immediate physical application.

Taking that in the chosen frame for representations with definite finite helicity $W_2=W_3=0$ and $W^0=S_1p^0$, $W_1=S_1p_1$, helicity operator can be defined as

$$\Lambda=S_1=\frac{W^0}{p^0}=\frac{W_1}{p_1}.$$

Widely used is the $SO(3)$-invariant form of the helicity operator $\Lambda=S_in_i$, where $n_i=p_i/p^0$ is the unit vector in direction of momentum. This expression reflects the sense of helicity as projection of the particle's spin on direction of its momentum. Since both W^m and p^m are Lorentz-covariant quantities, their proportionality should hold in any frame that reflects expression (4). However, as was mentioned in the Introduction, in the present approach it is problematic to get manifestly Lorentz-invariant expression for the proportionality coefficient itself.

2. HELICITY DESCRIPTION IN TWISTOR THEORY

In twistor theory [14, 15] it is well-known expression for helicity that is not only manifestly Poincare-invariant but is actually conformally-invariant. Let

$$Z^{\underline{\alpha}}=\begin{pmatrix}\omega^\alpha\\ \bar{u}_{\dot\alpha}\end{pmatrix}$$

be a twistor, ω^α and $\bar{u}_{\dot\alpha}$ are its $SL(2,C)$ spinor components and

$$\bar{Z}_{\underline{\alpha}}=\begin{pmatrix}\bar\omega^{\dot\alpha} & u_\alpha\end{pmatrix}=Z^+H,\ H=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

is a dual twistor. Twistor and its dual transform in **4** and $\bar{\mathbf{4}}$ representations of the $SU(2,2)$ group that is the four-fold covering of conformal group of Minkowski space-time. Then classical expression for helicity is given by

$$s=\frac{1}{2}Z^{\underline{\alpha}}\bar{Z}_{\underline{\alpha}}=\frac{1}{2}\left(\omega^\alpha u_\alpha+\bar\omega^{\dot\alpha}\bar{u}_{\dot\alpha}\right). \qquad (5)$$

In this Section we denote helicity by s because this is commonly used notation in twistor theory. To derive (5) from (1) and (4), one uses Cartan-Penrose representation for massless particle's momentum

$$p^m=-\bar{u}_{\dot\alpha}\tilde\sigma^{m\dot\alpha\alpha}u_\alpha \qquad (6)$$

and spinor form of its angular momentum tensor

$$M_{\alpha\beta\dot\alpha\dot\beta}=M_{mn}\sigma^m_{\alpha\dot\alpha}\sigma^n_{\beta\dot\beta}=\omega_{(\alpha}u_{\beta)}\varepsilon_{\dot\alpha\dot\beta}+\bar\omega_{(\dot\alpha}\bar{u}_{\dot\beta)}\varepsilon_{\alpha\beta}. \qquad (7)$$

We adhere to the same spinor algebra conventions as in [21].

In the first-quantized theory, commutation relations for twistors are

$$\left[Z^{\underline{\alpha}},\bar{Z}_{\underline{\beta}}\right]=\delta^{\underline{\alpha}}_{\underline{\beta}},\ \left[Z^{\underline{\alpha}},Z^{\underline{\beta}}\right]=\left[\bar{Z}_{\underline{\alpha}},\bar{Z}_{\underline{\beta}}\right]=0.$$

Similarly to operators of the space-time coordinates and momenta, they can be realized in the form of multiplication and differentiation operators

$$Z^{\underline{\alpha}},\ \bar{Z}_{\underline{\alpha}} = -\frac{\partial}{\partial Z^{\underline{\alpha}}}$$

that act in the space of functions on four-dimensional complex twistor space. There also exists another realization

$$Z^{\underline{\alpha}} = \frac{\partial}{\partial \bar{Z}_{\underline{\alpha}}},\ \bar{Z}_{\underline{\alpha}}$$

adapted to action on functions on the dual twistor space. Then Hermitian helicity operator takes the form

$$s = -\frac{1}{2}\left(Z^{\underline{\alpha}}\frac{\partial}{\partial Z^{\underline{\alpha}}} + 2\right)$$

or

$$s = \frac{1}{2}\left(\bar{Z}_{\underline{\alpha}}\frac{\partial}{\partial \bar{Z}_{\underline{\alpha}}} + 2\right).$$

Quantization of its eigenvalues is the consequence of integer-valuedness of the homogeneity degree of functions on (dual) twistor space.

3. DESCRIPTION OF PAULI-LUBANSKI PSEUDOVECTOR, HELICITY AND SPIN IN ORTHONORMAL TETRAD

3.1. ORTHONORMAL TETRAD IN MINKOWSKI SPACE

Conformally invariant description of helicity in twistor theory relies on the null momentum representation via two-component spinor and its complex conjugate. This null momentum can be identified with one of null column-vectors $n_m^{\{\pm 2\}}$ of attached to the massless particle's world line orthonormal tetrad $n_m^{[n]}$. Here index $[n]$=[0],[1],[2],[3] labels constituent four-vectors of the tetrad. In the light-cone basis to be defined below, it takes values {+2},{–2},[2],[3]. Vector components are labeled by index m. Such identification suggests that orthonormal tetrad can be used to obtain manifestly Lorentz-invariant expression for helicity operator of massless particles not only in the framework of twistor theory but also in the space-time description. Analogously, it can be used to derive Lorentz-invariant spin operator of massive particles from the Pauli-Lubanski pseudovector.

To substantiate the identification of massless particle's momentum with null component of the tetrad, below there are briefly reminded relevant properties of the tetrad and its realization via two-component spinors

$$n^{m}{}_{[m]} = \frac{1}{2}v^{\alpha}_{(\alpha)}\sigma^{m}_{\alpha\dot{\alpha}}\bar{v}^{\dot{\alpha}}_{(\dot{\alpha})}\tilde{\sigma}^{(\dot{\alpha})(\alpha)}_{[m]}. \qquad (8)$$

Note that expression (8) is a particular case of the general explicit representation for vector frame via Majorana spinors in Minkowski spaces of dimension D=2,3,4(mod8) proposed in [22]. Note also that 2×2 spinor matrices $v^{\alpha}_{(\alpha)}$ and $\bar{v}^{\dot{\alpha}}_{(\dot{\alpha})}$ are known as normalized dyads [15, 16]. They represent orthonormal bases in the spaces of dotted and undotted spinors and take value in SL(2,C) group because of the conditions

$$v^{\alpha}_{(\alpha)}v^{(\alpha)}_{\beta} = \delta^{\alpha}_{\beta},\ \bar{v}^{\dot{\alpha}}_{(\dot{\alpha})}v^{(\dot{\alpha})}_{\dot{\beta}} = \delta^{\dot{\alpha}}_{\dot{\beta}},$$

$$v^{\alpha}_{(\alpha)}v^{(\beta)}_{\alpha} = \delta^{(\beta)}_{(\alpha)},\ \bar{v}^{\dot{\alpha}}_{(\dot{\alpha})}\bar{v}^{(\dot{\beta})}_{\dot{\alpha}} = \delta^{(\dot{\beta})}_{(\dot{\alpha})}.$$

These spinor orthonormality conditions ensure that the tetrad is orthonormal

$$n^{n}{}_{[m]}n_{n}{}^{[k]} = \delta^{[k]}_{[m]}. \qquad (9)$$

Conditions (9) are defining relations of the orthonormal tetrad. They imply that the tetrad takes value in the orthogonal group SO(1,3). Like the dyad represents orthonormal basis in the spinor space, tetrad constitutes orthonormal basis in the tangent space to the space-time.

Let us note that the tetrade and spinor dyads are well established research tools in Gerenal relativity [16, 17]. Utility of the tetrade and dyad, as well as of their higher-dimensional generalizations, for formulating models of point particles and extended objects in string theory was appreciated by the mid of 1980s [23–25]. There they are known as Lorentz harmonics [26–38] that emphasizes their feature to take value in the (spinor covering of) the Lorentz group. Introduction of spinor Lorentz harmonics into space-time supersymmetric models of massless particle [30, 34, 35], tensionless string and p-branes [31–33, 39] as well as tensile string [36, 37, 22] and p-branes [38, 40, 41] allows to solve the problem of irreducible description of the κ-symmetry [42–44]. This is just one of applications of the Lorentz harmonics.

The key property of relations (9) is that they are invariant under $SO(1,3)_L \times SO(1,3)_R$ symmetry that is the product of two Lorentz groups. The first factor $SO(1,3)_L$ acts on the unbracketed indices and the second $SO(1,3)_R$ – on bracketed. Therefore introduction of the tetrad and dyads allows to retain the $SO(1,3)_L$ covariance and 'shift' symmetry breaking to the $SO(1,3)_R$ group.

From relations (9) it follows that the tetrad consists of one time-like and three space-like orthonormal vectors. For identification with null momentum it is necessary to introduce light-cone basis, in which two vectors of the tetrad are null

$$n_m^{\{+2\}} = \frac{1}{2}(n_m^{[0]} + n_m^{[1]}),\ n_m^{\{-2\}} = \frac{1}{2}(n_m^{[0]} - n_m^{[1]}).$$

They are labeled by weights under the $SO(1,1)_R$ subgroup of $SO(1,3)_R$ corresponding to rotations in the [0][1] plane. In the light-cone basis non-zero components of two-dimensional Minkowski metric and its inverse are

$$\eta^{\{+2\}\{-2\}} = \eta^{\{-2\}\{+2\}} = -\frac{1}{2},\ \eta_{\{+2\}\{-2\}} = \eta_{\{-2\}\{+2\}} = -2.$$

They connect covariant and contravariant two-dimensional vectors

$$n_{\{-2\}} = -2n^{\{+2\}},\ n_{\{+2\}} = -2n^{\{-2\}},$$

$$n^{\{+2\}} = -\frac{1}{2}n_{\{-2\}},\ n^{\{-2\}} = -\frac{1}{2}n_{\{+2\}}.$$

In the light-cone basis relativistic Pauli matrices $\tilde{\sigma}^{\{m\}(\dot{\alpha})(\alpha)}$ in (8) become projectors

$$\tilde{\sigma}^{\{+2\}} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},\ \tilde{\sigma}^{\{-2\}} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore two light-like vectors of the tetrad in terms of the normalized dyad acquire the form

$$n^{m\{+2\}} = -\frac{1}{2} v^{\alpha}_{(1)} \sigma^{m}_{\alpha\dot{\alpha}} \bar{v}^{\dot{\alpha}}_{(1)},\ n^{m\{-2\}} = -\frac{1}{2} v^{\alpha}_{(2)} \sigma^{m}_{\alpha\dot{\alpha}} \bar{v}^{\dot{\alpha}}_{(2)}. \quad (10)$$

Equalities (10) determine $SO(1,1)_R$ weights of the dyad components

$$v^{\alpha}_{(1)} = v^{\alpha +},\ v^{\alpha}_{(2)} = v^{\alpha -},\ \bar{v}^{\dot{\alpha}}_{(1)} = \bar{v}^{\dot{\alpha} +},\ \bar{v}^{\dot{\alpha}}_{(2)} = \bar{v}^{\dot{\alpha} -}.$$

Any of light-like vectors in (10) can be identified with the particle's momentum (6). For instance, identification $p^m = n^{m\{+2\}}$ implies identification (up to a sign) of projection parts of respective twistor and its dual with the components of the spinor bases

$$\sqrt{2} u^{\alpha} = -v^{\alpha}_{(1)},\ \sqrt{2} \bar{u}^{\dot{\alpha}} = -\bar{v}^{\dot{\alpha}}_{(1)}.$$

Then for other components of dyads one can also introduce a notation

$$v^{\alpha}_{(2)} = -\frac{1}{\sqrt{2}} v^{\alpha},\ \bar{v}^{\dot{\alpha}}_{(2)} = -\frac{1}{\sqrt{2}} \bar{v}^{\dot{\alpha}}$$

and the number factor is determined by the normalization conditions $u^{\alpha} v_{\alpha} = 1$ and $\bar{u}^{\dot{\alpha}} \bar{v}_{\dot{\alpha}} = 1$. Expressions in terms of introduced spinors for other components of the tetrad (8) in the light-cone basis acquire the form

$$n^{m\{-2\}} = \frac{1}{4} v^{\alpha} \sigma^{m}_{\alpha\dot{\alpha}} \bar{v}^{\dot{\alpha}},$$

$$n^{m[2]} = \frac{i}{2} \left(v^{\alpha} \sigma^{m}_{\alpha\dot{\alpha}} \bar{u}^{\dot{\alpha}} - u^{\alpha} \sigma^{m}_{\alpha\dot{\alpha}} \bar{v}^{\dot{\alpha}} \right), \quad (11)$$

$$n^{m[3]} = \frac{1}{2} \left(v^{\alpha} \sigma^{m}_{\alpha\dot{\alpha}} \bar{u}^{\dot{\alpha}} + u^{\alpha} \sigma^{m}_{\alpha\dot{\alpha}} \bar{v}^{\dot{\alpha}} \right).$$

Spinor bases can be used to decompose primary parts of twistor and its dual

$$\omega^{\alpha} = u^{\alpha} \omega_{(v)} + v^{\alpha} \omega_{(u)},\ \omega_{(u)} = -\omega^{\alpha} u_{\alpha},\ \omega_{(v)} = \omega^{\alpha} v_{\alpha},$$

$$\bar{\omega}^{\dot{\alpha}} = \bar{u}^{\dot{\alpha}} \bar{\omega}_{(v)} + \bar{v}^{\dot{\alpha}} \bar{\omega}_{(u)},\ \bar{\omega}_{(u)} = -\bar{\omega}^{\dot{\alpha}} \bar{u}_{\dot{\alpha}},\ \bar{\omega}_{(v)} = \bar{\omega}^{\dot{\alpha}} \bar{v}_{\dot{\alpha}}.$$

As a result, helicity (5) is presented in the form

$$s = -\frac{1}{2} \left(\omega_{(u)} + \bar{\omega}_{(u)} \right),$$

in which $SO(1,3)_L$ invariance is hidden but $SO(1,3)_R$ symmetry is manifestly broken.

3.2. PAULI-LUBANSKI PSEUDOVECTOR, HELICITY AND SPIN IN ORTHONORMAL TETRAD

From the orthonormality conditions (9) it follows that $|\det n^{m}_{[n]}| = 1$. Below there will be considered the orthonormal tetrad with value in the proper orthochronous Lorentz group, for which $\det n^{m}_{[n]} = 1$. Then from definition of the determinant

$$\det n = -\frac{1}{24} \varepsilon_{klmn} \varepsilon^{[k][l][m][n]} n^{k}_{[k]} n^{l}_{[l]} n^{m}_{[m]} n^{n}_{[n]} = 1,$$

we find that

$$\varepsilon^{klmn} = \varepsilon^{[k][l][m][n]} n^{k}_{[k]} n^{l}_{[l]} n^{m}_{[m]} n^{n}_{[n]}. \quad (12)$$

Equality (12) can be substituted into definition of the Pauli-Lubanski pseudovector (1) that gives

$$W^{k} = \frac{1}{2} n^{k}_{[k]} \varepsilon^{[k][l][m][n]} M_{[l][m]} p_{[n]}, \quad (13)$$

where

$$M_{[l][m]} = n^{l}_{[l]} n^{m}_{[m]} M_{lm},\ \ p_{[n]} = n^{n}_{[n]} p_{n}$$

satisfy the same relations of the $iso(1,3)$ algebra but with the bracketed indices.

In the frame, in which momentum of the massless particle equals $p^{m}_{0} = (E, E, 0, 0)$, $SO(1,3)$ symmetry is broken down to the subgroup $SO(1,1) \times SO(2)$. Here $SO(2)$ is the subgroup of Wigner little group $ISO(2)$ for massless representations of the Poincare group. $SO(1,1)$ group corresponds to Lorentz boosts in direction of the momentum p^{m}_{0}. After introduction of the tetrad, such a symmetry breaking is 'shifted' to breaking of the $SO(1,3)_R$ symmetry. It acts on the bracketed indices $n_{m}{}^{[n]} = (n_{m}{}^{\{n\}}, n_{m}{}^{[p]})$, where $\{m\} = \{\pm 2\}$ are $SO(1,1)_R$ weights and $[p] = 2,3$ corresponds to the $SO(2)_R$ vector index. Then Pauli-Lubanski pseudovector for massless particle with momentum p^m can be presented in the form

$$W^{k} = \frac{1}{2} \varepsilon^{\{k\}\{l\}} n^{k}{}_{\{k\}} p_{\{l\}} \varepsilon^{[p][r]} M_{[p][r]}. \quad (14)$$

To arrive at this expression it has been used the condition

$$p_{[r]} = p^{m} n_{m[r]} = 0$$

of the momentum orthogonality to space-like vectors of the tetrad. The momentum itself is identified with one of light-like vectors of the tetrad, e.g.

$$p^{m} = \rho^{\{-2\}} n^{m\{+2\}}, \quad (15)$$

where the Lagrange multiplier $\rho^{\{-2\}}$ has been introduced to maintain $SO(1,1)_R$ invariance. This implies that $p^{\{+2\}} = 0$ and $p^{\{-2\}} = -1/2\rho^{\{-2\}}$. Note that expression for momentum (15) can be interpreted as the Lorentz transformation from the frame, in which it equals p^{m}_{0}, to an arbitrary frame. It underlies Lorentz-harmonic formulation of massless superparticles [30] and null-super-p-branes [31–33, 39] in four-dimensional superspace. As a result of substitution of (15) into (14), expression for the Pauli-Lubanski pseudovector simplifies to

$$W^{k} = \frac{1}{2} M_{[2][3]} \rho^{\{-2\}} n^{k\{+2\}} = \frac{1}{2} M_{[2][3]} p^{k}. \quad (16)$$

Comparing (16) with (4), we get the expression for helicity

$$\Lambda = -\frac{1}{2\rho^{\{-2\}}} W^{m} n^{\{-2\}}_{m} = \frac{1}{2} M_{mn} n^{m}_{[2]} n^{n}_{[3]} \quad (17)$$

that is manifestly $SO(1,3)_L$ invariant.

For single-particle representations the form of operator M_{mn} is determined by value of their helicity. In classical mechanics of free massless particles M_{mn} is angular momentum tensor that is the Noether charge associated with the Lorentz invariance of the action. Only its spin part contributes to the Pauli-Lubanski pseudovector and its explicit form depends on considered realization of spin degrees of freedom. Upon first-quantization of the spin degrees of freedom this realization should match the expression for $so(1,3)$ generators in the representation theory.

Obtained expression (17) agrees with that for helicity in twistor theory (5), if one substitutes the angular momentum tensor (7) and realization of the orthonormal tetrad in terms of the spinor dyads (11).

Above consideration readily extends to massive particles. The stability group of the rest-frame momentum $p^{m}_{0} = (m, 0, 0, 0)$ is $SO(3)$ that is the little group in the

Wigner approach of induced representations. Particle's momentum is identified with the time-like component of the tetrad

$$p^m = mn^{m[0]}.$$

Similarly to the case of massless particle, this expression can be interpreted as the Lorentz transformation connecting the rest frame with an arbitrary frame. Such an identification induces $SO(1,3)_R$ symmetry breaking down to $SO(3)_R$. Then introduction of the tetrad components into definition (1) of the Pauli-Lubanski pseudovector using (12), again gives (13) but now

$$p^{[n]} = p^m n_m^{[n]}: \; p^{[0]} = m, \; p^{[i]} = 0, \; [i] = [1],[2],[3].$$

Therefore

$$W^k = \frac{m}{2}\varepsilon_{[i][j][k]} n^k{}_{[i]} M_{[j][k]},$$

where $M_{[j][k]} = n^m{}_{[j]} n^n{}_{[k]} M_{mn}$ satisfy the *so*(3) algebra relations. In terms of spin 3-vector

$$S_{[i]} = \frac{1}{2}\varepsilon_{[i][j][k]} M_{[j][k]}$$

the Pauli-Lubanski pseudovector acquires the form

$$W^k = mn^k{}_{[i]} S_{[i]}. \qquad (18)$$

In turn, spin 3-vector is expressed via the Pauli-Lubanski pseudovector as

$$S_{[i]} = \frac{1}{m} W^k n_{k[i]}. \qquad (19)$$

Like in the non-covariant approach, quartic Casimir operator for massive representations reduces to the Casimir of the *so*(3) algebra

$$W^2 = m^2 S^2. \qquad (20)$$

Expressions (18)-(20) are known to arise, e.g. in two-twistor mechanics of massive particles [45–47].

CONCLUSIONS

In this note it has been shown how introduction of the orthonormal tetrad in four-dimensional Minkowski space-time allows obtaining manifestly Lorentz-invariant expression for helicity of massless particle as projection of the Pauli-Lubanski pseudovector on the light-like tetrad component other than its momentum. For massive particle, its spin vector has been expressed as projection of the Pauli-Lubanski pseudovector on the space-like tetrad components orthogonal to momentum. Obtained expressions are general and model-independent and agree with those obtained in twistor approach to description of massless and massive particles.

It is of interest to study the possibility of extending application of orthonormal vector and spinor frames to description of particle's spin in other dimensions $D>4$ and to compare such a description in the space-time approach with that based on higher-dimensional generalizations of the (super)twistor approach [48, 49].

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REFERENCES

1. E. Wigner. Unitary representations of the inhomogeneous Lorentz group // *Ann. Math.* 1939, v. 40, p. 149-204; https://doi.org/10.2307/1968551

2. S. Weinberg. *The Quantum Theory of Fields. Vol.I. Foundations*. Cambridge: "Cambridge University Press", 1995, 634 p;
https://doi.org/10.1017/CBO9781139644167

3. I.L. Buchbinder, S.M. Kuzenko. *Ideas and Methods of Supersymmetry and Supergravity or A Walk Through Superspace*. Bristol: "Institute of Physics Publishing", 1998, 676 p.;
https://doi.org/10.1201/9780367802530

4. J.K. Lubański. Sur la theorie des particules élémentaires de spin quelconque. I // *Physica.* 1942, v. 9, p. 310-324;
https://doi.org/10.1016/s0031-8914(42)90113-7

5. J.K. Lubański. Sur la theorie des particules élémentaires de spin quelconque. II // *Physica.* 1942, v. 9, p. 325-338;
https://doi.org/10.1016/s0031-8914(42)90114-9

6. M. Chaichian, A.P. Demichev, N.F. Nelipa. The Casimir Operators of Inhomogeneous Groups // *Commun. Math. Phys*. 1983, v. 90, p. 353-372;
https://doi.org/10.1007/BF01206887

7. S.M. Kuzenko, A.E. Pindur. Massless particles in five and higher dimensions // *Phys. Lett. B.* 2021, v. 812, p. 136020;
https://doi.org/10.1016/j.physletb.2020.136020
arXiv:2010.07124 [hep-th];
https://doi.org/10.48550/arXiv.2010.07124

8. I.L. Buchbinder, S.A. Fedoruk, A.P. Isaev, M.A. Podoinitsyn. Massless finite and infinite spin representations of Poincare group in six dimensions // *Phys. Lett. B.* 2021, v. 813, p. 136064;
https://doi.org/10.1016/j.physletb.2021.136064
arXiv:2011.14725 [hep-th];
https://doi.org/10.48550/arXiv.2011.14725

9. I.A. Lahlali, N. Boulanger, A. Campoleoni. Coadjoint orbits of the Poincare group for discrete-spin particles in any Dimension // *Symmetry*. 2021, v. 13, p. 1749; https://doi.org/10.3390/sym13091749

10. V.I. Fushchich, I.Yu. Krivsky. On a possible approach to the variable-mass problem // *Nucl. Phys. B.* 1968, v. 7, p. 79-82;
https://doi.org/10.1016/0550-3213(68)90170-3
arXiv:quant-ph/0206057;
https://doi.org/10.48550/arXiv.quant-ph/0206057

11. V.I. Fushchich, A.G. Nikitin. *Symmetry of Equations of Quantum Mechanics.* M.: "Sciense", 1990, 400 p.

12. W.I. Fushchich, A.G. Nikitin. *Symmetries of Equations of Quantum Mechanics*. New York, Allerton Press, 1994, 480 p.

13. R. Marnelius, U. Martensson. Derivation of manifestly covariant quantum models for spinning relativistic particles // *Nucl. Phys. B.* 1990, v. 335, p. 395-420; https://doi.org/10.1016/0550-3213(90)90500-D

14. R. Penrose, M.A.H. MacCallum. Twistor theory: an approach to the quantisation of fields and space-time // *Phys. Rept.* 1972, v. 6, p. 241-316;
https://doi.org/10.1016/0370-1573(73)90008-2

15. R. Penrose, W. Rindler. *Spinors and space-time. Vol. II. Spinor and twistor methods in space-time geometry*. Cambridge: “Cambridge University Press”, 1986, 512 p.; https://doi.org/10.1017/CBO9780511524486

16. E.T. Newman, R. Penrose. An approach to gravitational radiation by a method of spin coeffcients // *J. Math. Phys.* 1962, v. 3, p. 896;
https://doi.org/10.1063/1.1724257

17. R. Penrose, W. Rindler. *Spinors and space-time. Vol.I. Two-spinor calculus and relativitsic fields*. Cambridge: “Cambridge University Press”, 1984, 472 p.;
https://doi.org/10.1017/CBO9780511564048

18. E.P. Wigner. Relativistische Wellengleichungen // *Z. Physik*. 1948, v. 124, p. 665-684;
https://doi.org/10.1007/BF01668901

19. X. Bekaert, E.D. Skvortsov. Elementary particles with continuous spin *// Int. J. Mod. Phys. A.* 2017, v. 32, 1730019;
https://doi.org/10.1142/S0217751X17300198
arXiv:1708.01030 [hep-th];
https://doi.org/10.48550/arXiv.1708.01030

20. X. Bekaert, N. Boulanger. The unitary representations of the Poincare group in any spacetime dimension // *SciPost Phys. Lect. Notes.* 2021, v. 30, p. 1;
https://doi.org/10.21468/SciPostPhysLectNotes.30
arXiv:hep-th/0611263;
https://doi.org/10.48550/arXiv.hep-th/0611263

21. J. Wess, J. Bagger. *Supersymmetry and Supergravity*. Princeton: “Princeton University Press”, 1992, 259 p.

22. I.A. Bandos, A.A. Zheltukhin. Twistor-like approach in the Green-Schwarz *D*=10 superstring theory // *Phys. Part. Nucl*. 1994, v. 25, p. 453-477.

23. D.V. Volkov, A.A. Zheltukhin. String description in the space and superspace // *Ukr. Phys. J*. 1985, v. 30, p. 809-813.

24. E. Sokatchev. Light-cone harmonic superspace and its applications // *Phys. Lett. B*. 1986, v. 169, p. 209-214;
https://doi.org/10.1016/0370-2693(86)90652-0

25. E. Sokatchev. Harmonic superparticle // *Class. Quantum Grav*. 1987, v. 4, p. 237-246;
https://doi.org/10.1088/0264-9381/4/2/007

26. E. Nissimov, S. Pacheva, S. Solomon. Covariant first and second quantization of the *N*=2, *D*=10 Brink-Schwarz superparticle // *Nucl. Phys. B.* 1988, v. 296, p. 462-492;
https://doi.org/10.1016/0550-3213(88)90681-5

27. E. Nissimov, S. Pacheva, S. Solomon. Covariant canonical quantization of the Green-Schwarz superstring // *Nucl. Phys. B.* 1988, v. 297, p. 349-373;
https://doi.org/10.1016/0550-3213(88)90024-7

28. R.E. Kallosh, M.A. Rakhmanov. Covariant quantization of the Green-Schwarz superstring // *Phys. Lett. B*. 1988, v. 209, p. 233-238;
https://doi.org/10.1016/0370-2693(88)90938-0

29. Y. Eisenberg, S. Solomon. The twistor geometry of the covariantly quantized Brink-Schwarz superparticle // *Nucl. Phys. B*. 1988, v. 309, p. 709-732;
https://doi.org/10.1016/0550-3213(88)90337-9

30. I.A. Bandos. Superparticle in Lorentz-harmonic superspace // *Sov. J. Nucl. Phys.* 1990, v. 51, p. 906-914.

31. I.A. Bandos, A.A. Zheltukhin. Twistors, harmonics, and zero-super-*p*-branes // *J. Experiment. Theor. Phys. Lett*. 1990, v. 51, p. 618-621.

32. I.A. Bandos, A.A. Zheltukhin. Null super *p*-brane: Hamiltonian dynamics and quantization // *Phys. Lett. B.* 1991, v. 261, p. 245-250;
https://doi.org/10.1016/0370-2693(91)90322-H

33. I.A. Bandos, A.A. Zheltukhin. Covariant quantization of null supermembranes in four-dimensional spacetime // *Theor. Math. Phys.* 1991, v. 88, p. 925-937;
https://doi.org/10.1007/BF01027695

34. F. Delduc, A.S. Galperin, E. Sokatchev. Lorentz-harmonic (super)fields and (super)particles // *Nucl. Phys. B.* 1992, v. 368, p. 143-171;
https://doi.org/10.1016/0550-3213(92)90201-L

35. A.S. Galperin, P.S. Howe, K.S. Stelle. The superparticle and the Lorentz group // *Nucl. Phys. B.* 1992, v. 368, p. 248-280;
https://doi.org/10.1016/0550-3213(92)90527-I
arXiv:hep-th/9201020;
https://doi.org/10.48550/arXiv.hep-th/9201020

36. I.A. Bandos, A.A. Zheltukhin. Spinor Cartan moving *n*-hedron, Lorentz-harmonic formulations of superstrings, and *κ*-symmetry // *J. Experiment. and Theor. Phys. Lett.* 1991, v. 54, p. 421-424.

37. I.A. Bandos, A.A. Zheltukhin. Green-Schwarz superstrings in spinor moving frame formalism // *Phys. Lett. B.* 1992, v. 288, p. 77-84;
https://doi.org/10.1016/0370-2693(92)91957-B

38. I.A. Bandos, A.A. Zheltukhin. Generalization of Newman-Penrose dyads in connection with the action integral for supermembranes in an 11-dimensional space // *J. Experiment. and Theor. Phys. Lett.* 1992, v. 55, p. 81-84.

39. I.A. Bandos, A.A. Zheltukhin. Null super *p*-branes quantum theory in 4-dimensional space-time // *Fortschr. Phys.* 1993, v. 41, p. 619-676;
https://doi.org/10.1002/prop.19930410703

40. I.A. Bandos, A.A. Zheltukhin, Lorentz harmonics and new formulations of the superstrings in *D*=10 and supermembranes in *D*=11 // *Phys. Atom. Nucl.* 1993, v. 56, p. 113-121.

41. I.A. Bandos, A.A. Zheltukhin. *N*=1 super-*p*-branes in twistor-like Lorentz harmonic formulation // *Class. Quantum Grav*. 1995, v. 12, p. 609-626;
https://doi.org/10.1088/0264-9381/12/3/002
arXiv:hep-th/9405113;
https://doi.org/10.48550/arXiv.hep-th/9405113

42. W. Siegel. Hidden local supersymmmetry in the supersymmetric particle action // *Phys. Lett. B*. 1983, v. 128, p. 397-399;
https://doi.org/10.1016/0370-2693(83)90924-3

43. J. de Azcarraga, J. Lukierski. Supersymmetric particles with internal symmetries and central charges // *Phys. Lett. B*. 1982, v. 113, p. 170-174;
https://doi.org/10.1016/0370-2693(82)90417-8

44. M.B. Green, J.H. Schwarz. Covariant description of superstrings // *Phys. Lett. B*. 1984, v. 136, p. 367-370; https://doi.org/10.1016/0370-2693(84)92021-5

45. Z. Perjes. Twistor variables of relativistic mechanics // *Phys. Rev. D*. 1975, v. 11, p. 2031-2041;
https://doi.org/10.1103/PhysRevD.11.2031

46. A. Bette. On a point-like relativistic and spinning particle // *J. Math. Phys*. 1984, v. 25, p. 2456-2460; https://doi.org/10.1063/1.526463

47. J.A. de Azcarraga, S. Fedoruk, J.M. Izquierdo, J. Lukierski. Two-twistor particle models and free massive higher spin fields // *J. High Energy Phys.* 2015, v. 1504, p. 010; https://doi.org/10.1007/JHEP04(2015)010 arXiv:1409.7169 [hep-th]; https://doi.org/10.48550/arXiv.1409.7169

48. D.V. Uvarov. Ambitwistors, oscillators and massless fields on AdS_5 // *Phys. Lett. B.* 2016, v. 762, p. 415-420; https://doi.org/10.1016/j.physletb.2016.09.065 arXiv:1607.05233 [hep-th]; https://doi.org/10.48550/arXiv.1607.05233

49. D.V. Uvarov. Supertwistor formulation for higher-dimensional superstrings // *Class. Quantum Grav*. 2007, v. 24, p. 5383-5400; https://doi.org/10.1088/0264-9381/24/22/004 arXiv:hep-th/0703051; https://doi.org/10.48550/arXiv.hep-th/0703051

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ПСЕВДОВЕКТОР ПАУЛІ-ЛЮБАНСЬКОГО ТА ОРТОНОРМОВАНА ТЕТРАДА

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Як відомо, власні значення спіральності – величини, що визначаються як коефіцієнт пропорційності між псевдовектором Паулі-Любанського та імпульсом, – використовуються для класифікації безмасових унітарних представлень групи Пуанкаре. Однак представляє проблему отримання виразу для неї з явною лоренц-інваріантністю у будь-який інший спосіб, окрім твісторного підходу. Показано, що введення ортонормованої тетради у просторі-часі Мінковського дає можливість здобути такий вираз у термінах псевдовектора Паулі-Любанського. Введення ортонормованої тетради також дозволяє виразити у лоренц-інваріантний спосіб тривимірний вектор спіну масивної частинки через псевдовектор Паулі-Любанського.