INTERACTION OF RELATIVISTIC PARTICLES WITH CRYSTALS AND MATTER

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A SPECIAL FEATURE OF ELECTRO-ATOMIC AND ELECTRO-NUCLEAR REACTIONS IN CRYSTALS AT HIGH ENERGIES

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In the present paper it is proved the following theorem: "All possible processes of interaction relativistic electrons (positrons) with crystal, which lie in fact that at first relativistic electron (positron) radiates virtual photon, and then virtual photon produces photo-atomic or photo-nuclear reaction, can be treated as two consecutive processes: 1 – coherent bremsstrahlung; 2 – photo-atomic or photo-nuclear reactions".

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INTRODUCTION

First of all we must find wave function of relativistic electron (positron) $\Psi(\vec{r},t)$ in crystal potential $U(\vec{r})$, i. e. find solution of Dirac equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t)=\hat{H}\,\Psi(\vec{r},t), \qquad (1)$$

where Hamilton operator

$$\hat{H}=c\hat{\vec{\alpha}}\hat{\vec{p}}+\hat{\beta}mc^2\pm e\,U(\vec{r}). \qquad (2)$$

Hamilton operator (2) does not depend on time. In this case wave function (1) has the form: $\Psi(\vec{r},t)=\Psi(\vec{r})\,e^{-iEt/\hbar}$, where E – is the energy of electron (positron).

Then we have:

$$\hat{H}\,\Psi_n(\vec{r})=E_n\Psi_n(\vec{r}), \qquad (3)$$

where $\Psi_n(\vec{r})$ and E_n – are own functions and own numbers of Hamilton operator $\hat{H}$.

In our case we have the thing crystal target which surface coincide with one of the main crystallographic plane. Crystal potential $U(\vec{r})=U(\vec{r}+\vec{L})$, where w $\vec{L}$ – are vectors of crystal lattice. Let us indicate operator of transmissions $\hat{T}_{\vec{L}}$ by equation $\hat{T}_{\vec{L}}\,f(\vec{r})=f(\vec{r}+\vec{L})$ and apply it to the equation (3):

$$\hat{T}_{\vec{L}}[\hat{H}\,\Psi_n(\vec{r})]=\hat{T}_{\vec{L}}[E_n\Psi_n(\vec{r})]=E_n\Psi_n(\vec{r}+\vec{L}),$$

$$\hat{H}\,[\hat{T}_{\vec{L}}\Psi_n(\vec{r})]=E_n[\hat{T}_{\vec{L}}\Psi_n(\vec{r})]=E_n\Psi_n(\vec{r}+\vec{L}). \qquad (4)$$

From equations (4) it is evident that operators $\hat{H}$ and $\hat{T}_{\vec{L}}$ – are commutative. Hamilton operator $\hat{H}$ is invariant relatively all $\hat{T}_{\vec{L}}$. All functions $\Psi_n(\vec{r}+\vec{L})$ are own functions of Hamilton operator $\hat{H}$ with the same own numbers E_n.

F. Bloch [1] established that in consequence of such properties of Hamilton operator $\hat{H}$ (see Eq. (4)) wave function of electron in crystal looks like a product of plain wave and periodical function with period equal to the crystal lattice period. Periodical function can be represented by Fourier row, i. e. relativistic electron wave function in crystal can be presented in the form [2]:

$$\Psi(\vec{r},t)=e^{-iEt/\hbar}\sum_{\vec{g}}C_{\vec{g}}\;\bar{u}(\vec{p}+\vec{g},s)\;e^{i(\vec{p}+\vec{g})\vec{r}/\hbar}, \qquad (5)$$

where $\vec{g}$ – are vectors of reciprocal lattice of crystal. Formula (5) can be obtained as a solution of Dirac equation (1) using Born approximations [3] as well.

As a result of such type of relativistic electron wave function in the crystal media (see. Eq. (5)) calculating probabilities of any electro-magnetic process in crystal we obtain it with multiplier $\sum_{\vec{g}}\delta(\vec{q}-\vec{g})$, where $\vec{q}$ is momentum transferred to whole crystal. This sum of δ-functions is expressed the law of momentum. conservation. One can see, that transferred momentum must be equal one of the reciprocal lattice vectors $\vec{g}$. Such result may be ascribed to coherent scattering relativistic electrons in the crystal media.

Let us next consider the following theorem:

"All possible processes of interaction relativistic electrons (positrons) with crystal, which lie in fact that at first relativistic electron (positron) radiates virtual photon, and then virtual photon produces photo-atomic or photo-nuclear reaction, can be treated as two consecutive processes: 1 – coherent bremsstrahlung; 2 – photo-atomic or photo-nuclear reactions".

EVIDENCE OF THEOREM

By way of example let us consider the process of ionization of crystal atom by relativistic electron (positron).

The transition of relativistic electron (positron) from initial state $\Psi_i(x)$ into final state $\Psi_f(x)$ creates a ***current***, which must be expressed by the matrix element:

$$j^{\mu}(x)=\pm ec\;\overline{\Psi}_f(x)\;\hat{\gamma}^{\mu}\;\Psi_i(x), \qquad (6)$$

where $\pm e$ is a charge of electron (positron), bispinos $\Psi_i(x)$ and $\Psi_f(x)$ are solutions of Dirac equation for relativistic electron (positron) in the crystal potential [3],

where 4-vector: x is $\vec{r} = \{x_1, x_2, x_3\}$, $ict = x_4$. Dirac matrices $\hat{\gamma}^\mu$ are of the form of:

$$\hat{\gamma}^\mu = -i\,\hat{\beta}\,\hat{\alpha}^\mu = i\,\hat{\alpha}^\mu\,\hat{\beta} = i\begin{pmatrix} 0 & -\hat{\sigma}^\mu \\ \hat{\sigma}^\mu & 0 \end{pmatrix} \quad \text{for}$$

$\mu = 1, 2, 3$, where $\hat{\sigma}^\mu$ are corresponding Pauli matrices; $\hat{\gamma}^4 = \hat{\beta} = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix}$; $\hat{\alpha}^\mu = \begin{pmatrix} 0 & \hat{\sigma}^\mu \\ \hat{\sigma}^\mu & 0 \end{pmatrix}$;

and $\overline{\Psi}(x) = i\Psi^+(x)\hat{\beta}$. By this means the current of transition relativistic electron (positron) from initial state $\Psi_i(x)$ into final state $\Psi_f(x)$ (see Eq. (6)) can be presented in the form of 3-dimensional vector $\vec{j}(x) = \pm ec\,\Psi_f^+(x)\,\hat{\vec{\alpha}}\,\Psi_i(x)$ with components $j^\mu(x) = \pm ec\,\Psi_f^+(x)\,\hat{\alpha}^\mu\,\Psi_i(x)$ for $(\mu = 1,2,3)$ and $j^4(x) = ic\rho = \pm iec\Psi_f^+(x)\,\hat{\beta}\,\Psi_i(x)$, where ρ – is density of charge of relativistic electron (positron).

The current of transition of relativistic electron (positron) from initial state $\Psi_i(x)$ into final state $\Psi_f(x)$ creates ***electromagnetic field*** with 4-vector potential $A^\mu(x)$, where $\vec{A}(x) = \{A_{(1)}(x), A_{(2)}(x), A_{(3)}(x)\}$ is usual 3-dimensional vector potential, and $A_{(4)}(x) = i\varphi(x)$, where $\varphi(x)$ usual scalar potential. The process of excitation electromagnetic field by the current (6) describes by D'Alamberts equation, which is the extension of Maxwell equations:

$$\underline{\Pi}\,A^\mu(x) = -\frac{4\pi}{c}\,j^\mu(x), \qquad (7)$$

where $\underline{\Pi}$ – D'Alamberts operator. In order to found a solution equation (7) we introduce a Green function, or propagator. ***Propagator*** $G(x-y)$ is determined by equation:

$$\underline{\Pi}\,G(x-y) = \delta^4(x-y), \qquad (8)$$

and its Fourier transformation is:

$$G(x-y) = \int \frac{d^4q}{(2\pi)^4}\,e^{-iq(x-y)}G(q^2), \qquad (9)$$

where $G(q^2) = -1/q^2$ for $q^2 \neq 0$. With the aim to allow for receiving correct behavior of propagator in the pole $q^2 = 0$ we added an infinitely small imaginary quantity, i. e.

$$G(x-y) = \int \frac{d^4q}{(2\pi)^4}\,e^{-iq(x-y)}\left(\frac{-1}{q^2 - i\varepsilon}\right). \qquad (10)$$

So the ***potential*** $A^\mu(x)$, given by equation (7), is

$$A^\mu(x) = \int d^4y\,G(x-y)\;j^\mu(y). \qquad (11)$$

By action of potential $A^\mu(x)$ an atomic electron transits from initial bound state, which describes by the wave function $\varphi_i(x)$, into the final free state, which describes by the wave function $\varphi_f(x)$. This process describes by the first order S-matrix:

$$S^{(1)}_{i\to f} = -ie\int d^4x\;\overline{\varphi}_f(x)\;\hat{\gamma}\,\hat{A}(x)\,\varphi_i(x). \qquad (12)$$

It follows that the process, which we investigate can be described by the second order S-matrix:

$$S^{(2)}_{i\to f} = \frac{4\pi i}{c}\iint d^4x\,d^4y\left[e\,\overline{\phi}_f(x)\,\hat{\gamma}_\mu\phi_i(x)\right]\times \qquad (13)$$
$$\times G(x-y)\;j^\mu(y).$$

Taking into account specific view of $j^\mu(y)$ function, which is given in formula (6), second order ***S-matrix*** of investigated process (13) can be represented by 12-multiple integral over $d^4x\,d^4y\,d^4q$:

$$S^{(2)}_{i\to f} = \frac{4\pi i}{c}\iint d^4x\,d^4y\left[e\,\overline{\varphi}_f(x)\,\hat{\gamma}_\mu\varphi_i(x)\right]G(x-y)\,\delta_{\mu\nu}\,[\pm ec\,\overline{\psi}_f(y)\;\hat{\gamma}_\nu\,\psi_i(y)\,]. \qquad (14)$$

Integration in (14) can be executed in an arbitrary order.

Let us consider propagator (see Eq.10) in some detail. It is evident that in formula (10)

$$q(x-y) = \vec{k}(\vec{r} - \vec{r}') + iq_{(4)}c(t-t'),\;\; q^2 = \vec{k}^2 - q^2_{(4)} = \frac{\omega^2}{c^2} - q^2_{(4)},\;\; d^4q = d^3k\;dq_{(4)}.$$

At first let us calculate the integral over a close outline C in the complex-valued plane $q_{(4)}$:

$$\int_C \frac{dq_{(4)}}{2\pi}\,e^{-iq_{(4)}ct}\left(\frac{-1}{\omega^2/c^2 - q^2_{(4)} - i\varepsilon}\right) = -\frac{ic}{2|\omega|}\,e^{-i|\omega||t|}. \qquad (15)$$

The integral (see Eq. (15)) has poles of first order in the points: $q_{(4)} = \pm\frac{\omega}{c}\left(1 - \frac{i\varepsilon c^2}{2\omega^2}\right)$. If $t > 0$, then contour of integrating must be supplemented by semicircular with radius, which is going into infinity in the lower half-plane of complex plane, where $\mathrm{Im}(q_{(4)}) < 0$. The integral is equal to the deduction $2\pi i\,\mathrm{Re}\,s\left(\frac{\omega}{c} - i0\right)$, which is consistent with formula (15). If $t < 0$, then contour of integrating must be supplemented by semicircular with radius, which is going

into infinity in the upper half-plane of complex plane, where $\mathrm{Im}(q_{(4)}) > 0$. The integral is equal to the deduction $2\pi i \,\mathrm{Re}\,s\left(-\frac{\omega}{c} + i0\right)$, which is consistent with formula (15) too.

Taking into account (15), ***propagator function*** (10) can be represented as:

$$G(\vec{r}-\vec{r}^{\,/},\ t-t^{/}) = \frac{-ic}{2(2\pi)^3}\int \frac{d^3k}{|\omega|}\, e^{-i[\vec{k}(\vec{r}-\vec{r}^{\,/})-(|\omega|\,|t-t^{/}|)]}\ . \tag{16}$$

Operators of potentials of electromagnetic field $\hat{A}^{\mu}(x)$ with indexes $\mu = \{1,2,3\}$, which are consistent with processes of giving rise to photon in the point of space $\vec{r}$ in the time moment t and its absorption in the point of space $\vec{r}^{\,/}$ in the time moment $t^{/}$ can be shown up as:

$$\hat{\vec{A}}(\vec{r},t) = \sqrt{\frac{2\pi}{\omega}}\ \hat{c}_{\vec{k}\lambda}\vec{\varepsilon}_{\lambda}e^{-i(\vec{k}\vec{r}-\omega t)},$$

$$\hat{\vec{A}}^{+}(\vec{r}^{\,/},t^{/}) = \sqrt{\frac{2\pi}{\omega}}\ \hat{c}^{+}_{\vec{k}\lambda}\vec{\varepsilon}_{\lambda}e^{i(\vec{k}\vec{r}^{\,/}-\omega t^{/})}, \tag{17}$$

where $\omega = c\left|\vec{k}\right|$, $\hat{c}_{\vec{k}\lambda}$ and $\hat{c}^{+}_{\vec{k}\lambda}$ – operators of giving rise and absorption photon with energy ω, wave vector $\vec{k}$ and vector of polarization $\vec{\varepsilon}_{\vec{k}\lambda}$ $(\lambda = 1,2)$. In formulas (17) we us Coulomb calibration $div\vec{A} = 0$.

It is easily to notice, that function of propagator (16) can be represented as coagulation of operators (17):

$$G(\vec{r}-\vec{r}^{\,/},\ t-t^{/}) = \frac{-ic}{(2\pi)^3}\int d^3k\ \hat{\vec{A}}(\omega,\vec{k},\lambda,\vec{r},t)\ \hat{\vec{A}}^{+}(\omega,\vec{k},\lambda^{/},\vec{r}^{\,/},t^{/})\,. \tag{18}$$

In the case of potential of electromagnetic field of virtual photon $\hat{A}^{(4)}(x)$ we us formulas analogous to (16), (17), but with vector of polarization $\vec{\varepsilon}_{\vec{k},4} = \vec{k}/\left|\vec{k}\right|$. By this means the ***propagator*** (10) shows up as:

$$G(x-y)\,\delta_{\mu\nu} = \frac{-ic}{(2\pi)^3}\int d^3k\ \hat{A}_{\mu}(x)\ \hat{A}^{+}_{\nu}(y)\,. \tag{19}$$

Substituting (19) in (Eq. (14)), shows that S-matrix of investigated process is:

$$S^{(2)}_{i\to f} = \frac{4\pi i}{c}\int d^3k\ [ec\int d^4x\,\overline{\varphi}_f(x)\,\hat{\gamma}_{\mu}\hat{A}_{\mu}(k,x)\,\varphi_i(x)]\ [\pm ec\int d^4y\,\overline{\psi}_f(y)\,\hat{A}^{+}_{\nu}(k,y)\,\hat{\gamma}_{\nu}\,\psi_i(y)\ ]\ , \tag{20}$$

where in brackets […] stand two S-matrices of first order, which describe 2 processes – photon radiation by relativistic electron (positron) in crystal and photoionization of crystal atom (see Eq. (12)).

Integrating (20) with respect to the $dx_{(4)}\,dy_{(4)}$, i.e. with respect to the time t and to the $t^{/}$, we obtain S-matrix of ionization of crystal atom by relativistic electron in the form of package:

$$S^{(EI)}_{i\to f} = \frac{4\pi i}{c}\int d^3k\ [\pm M^{(BS)}_{i\to f}\,\delta(E_i - E_f - \hbar\omega)]\ [M^{(PI)}_{i\to f}\,\delta(\hbar\omega + \varepsilon_i - \varepsilon_f)\ ]\ , \tag{21}$$

where $S^{(BS)}_{i\to f} = \pm M^{(BS)}_{i\to f}\,\delta(E_i - E_f - \hbar\omega)$ and $S^{(PI)}_{i\to f} = M^{(PI)}_{i\to f}\,\delta(\hbar\omega + \varepsilon_i - \varepsilon_f)$ are S-matrices of bremsstrahlung and photoionization processes. $\pm M^{(BS)}_{i\to f}$ and $M^{(PI)}_{i\to f}$ are matrices elements of bremsstrahlung and photoionization processes, which obtained as a result of integrating over

$$d\,x_{(1)}d\,x_{(2)}\,d\,x_{(3)} = d^3r \qquad \text{and}$$

$$d\,y_{(1)}d\,y_{(2)}\,d\,y_{(3)} = d^3r^{/}\,.$$

If we represent d^3k in formula (21) in the form of $d^3k = k^2 d\Omega\, dk = c^{-3}\omega^2 d\omega\, d\Omega$, and vector $\vec{k} = \hat{k}\omega/c$, where $\hat{k}$ – is the unit vector in the direction of virtual photon and integrated over $d\,\omega$, we obtain:

$$S^{(EI)}_{i\to f} = M^{(EI)}_{i\to f}\,\delta(E_i - E_f + \varepsilon_i - \varepsilon_f)\,. \tag{22}$$

The ***probability*** of anyone quantum transition proportional the square of S-matrix:

$$W = \left|S_{i\to f}\right|^2 \prod_j \frac{V_j d^3p_j}{(2\pi)^{3j}}\,. \tag{23}$$

The probability of investigated process is:

$$W^{(EI)} = \left|M^{(EI)}_{i\to j}\right|^2 \delta(E_i - E_f + \varepsilon_i - \varepsilon_f)\frac{d^3P_f\, d^3p_f}{(2\pi)^5}\,. \tag{24}$$

We call attention to the formulas (20), (21), where wave functions of relativistic electron (positron) in a crystal $\Psi_i(x)$ and $\Psi_f(x)$ according to the Bloch theorem must be products of plane wave and periodical function with period of crystal lattice. By this means these functions are the packages of plane waves (see formula 5), where $\vec{g}$ are vectors of reciprocal lattice of crystal.

Using such functions in formula (21) we obtain:

$$S_{i\to f}^{(BS)} = \pm M_{i\to f}^{(BS)}\,\delta(E_i - E_f - \hbar\omega) = \pm Q_{i\to f}\sum_{\vec{g}}\,\delta(\vec{P}_i - \vec{P}_f - \vec{k} - \vec{g})\,\delta(E_i - E_f - \hbar\omega)\,. \qquad (25)$$

By this means formula (21) must appear as:

$$S_{i\to f}^{(EI)} = \frac{4\pi i}{c}\int d^3k\,[\pm Q_{i\to f}\sum_{\vec{g}}\delta(\vec{P}_i - \vec{P}_f - \vec{k} - \vec{g})\,\delta(E_i - E_f - \hbar\omega)]\;[M_{i\to f}^{(PI)}\,\delta(\hbar\omega + \varepsilon_i - \varepsilon_f)\,]\;. \qquad (26)$$

One can see that we have discrete multitude of vectors $\vec{g}$ in the expression of δ-function $\delta(\vec{P}_i - \vec{P}_f - \vec{k} - \vec{g})$, which is the law of momentum conservation. The availability of δ – functions $\delta(\vec{P}_i - \vec{P}_f - \vec{k} - \vec{g})$ in formula (see. Eq. (26)), which we represent in shorthand symbol as $\delta(\vec{k} - \vec{q})$, where $\vec{q} = \vec{P}_i - \vec{P}_f - \vec{g}$, ***makes possible to use identity:***

$$\left|\int d^3k\,F(\vec{k})\,\delta(\vec{k} - \vec{q})\right|^2 = \int d^3k\left|F(\vec{k})\right|^2\delta(\vec{k} - \vec{q})\,, \qquad (27)$$

where $F(\vec{k})$ is equal of expression under the integral in formula (26).

In general, operations of integrating and modulus square are not commutative. But in the presence of δ-functions $\delta(\vec{k} - \vec{q})$ under the integral it becomes possible, and these ***two operations are commutative***. One can to recognize that in our case integrated probability of considered process can be presented in the form of package:

$$W^{(EI)} = \sum_{\vec{g}}\;\int d^3k\,W^{(CBS)}(\vec{k},\omega,\vec{g})\cdot W^{(PI)}(\vec{k},\omega), \qquad (28)$$

where $W^{(CBS)}(\vec{k},\omega,\vec{g})$ is differential over the intermediate states probability of coherent bremsstrahlung of relativistic electron (positron) in crystal [4], and $W^{(PI)}(\vec{k},\omega)$ is differential over the intermediate states probability of photo-ionization of crystal atom. By this means that under investigation process ***can be treated as two consecutive processes.***

CONCLUSIONS

It is clear that formula (28) remains true if we in the place of probability $W^{(PI)}(\vec{k},\omega)$ use probability of any one photo-atomic or photo-nuclear process. Furthermore, the contribution of virtual photons with longitudinal polarization in the case of high energy electrons (positrons) is negligible small, i. e. virtual photons and real photons are not distinguishable.

By this means the statement that: ***"All possible processes of interaction relativistic electrons (positrons) with crystal, which lie in fact that at first relativistic electron (positron) radiates virtual photon, and then virtual photon produces photo-atomic or photo-nuclear reaction, can be treated as two consecutive processes: 1 – coherent bremsstrahlung; 2 – photo-atomic or photo-nuclear reactions"*** is true. (The theorem is proved).

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ОСОБЛИВІСТЬ ЕЛЕКТРОАТОМНИХ ТА ЕЛЕКТРОЯДЕРНИХ РЕАКЦІЙ В КРИСТАЛАХ ПРИ ВИСОКИХ ЕНЕРГІЯХ

В.Л. Мороховський

Доведена наступна теорема: «Всі можливі процеси взаємодії релятивістських електронів (позитронів) із кристалом, які полягають у тому, що спочатку релятивістський електрон (позитрон) випромінює віртуальний фотон, а потім віртуальний фотон викликає фотоатомну або фотоядерну реакцію, – можуть розглядатися як два послідовні процеси: 1 – когерентне гальмівне випромінювання; 2 – фотоатомна або фотоядерна реакція».