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FRICTION AND DIFFUSION COEFFICIENTS IN A RELATIVISTIC ELECTRON BEAM MOVING IN AN UNDULATOR

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The influence of the incoherent spontaneous emission field of relativistic electrons moving in the spatially periodic magnetic field of an undulator on the change in the mean value and rms longitudinal momentum of the electrons is considered. The friction and diffusion coefficients are obtained, and their dependencies on both the distance along the undulator and the initial electron momentum spread are determined.

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INTRODUCTION

As is known, charged particles moving with acceleration radiate electromagnetic waves (see, for example, [1]). When a beam of charged particles moves in an external periodic field (undulator), an individual charged particle is located in the incoherent fields of other particles radiation, and will experience the action of random forces from other particles of the beam This will lead to a change in the average momentum value and particle diffusion in momentum space [2, 3].

The investigation of these processes is of interest to clarify the necessary conditions for a self-amplification of spontaneous emission by an ultrarelativistic electron beam passing through an undulator [4–6].

In this paper the motion of a relativistic electron beam in the spatially periodic magnetic field of an undulator is considered, taking into account the influence of incoherent electromagnetic fields on this motion. Expressions have been obtained for the friction and diffusion coefficients resulting from the fields of spontaneous undulator radiation emitted by beam electrons.

1. FORMULATION OF THE PROBLEM

We consider a relativistic electron beam moving in the positive direction of the z axis ($z \geq 0$) in a helical magnetic field $\mathbf{H}_\mathrm{u} = \mathrm{H}_0\left[\mathbf{e}_\mathrm{x}\cos(\mathrm{k_u z}) + \mathbf{e}_\mathrm{y}\sin(\mathrm{k_u z})\right]$, where $k_u = 2\pi/\lambda_u$, H_0 and λ_u are the amplitude and period of the magnetic field, respectively; $\boldsymbol{e}_{x,y}$ are the unit vectors of the Cartesian coordinate system x, y, z.

The equations of motion of an electron can be written as

$$\frac{d}{dt}\boldsymbol{p} = \sum_{\boldsymbol{s}} \boldsymbol{F^{(s)}}\,[x(t), t], \qquad (1)$$

$$\boldsymbol{F}^{(s)}(x,t) = e\{\boldsymbol{E}_s(x,t) + [\boldsymbol{\beta}\boldsymbol{H}_s(x,t)]\}, \qquad (2)$$

where $\boldsymbol{F}^{(s)}$,is the pair interaction force of electrons, $x = (\boldsymbol{r}, \boldsymbol{p})$, $\boldsymbol{p} = m\gamma\boldsymbol{v}$, $\gamma = \left(1-\beta^2\right)^{-1/2}$, $\boldsymbol{\beta} = \boldsymbol{v}/c$, $\boldsymbol{E}_s, \boldsymbol{H}_s$ are the electric and magnetic fields produced by the s-th electron in undulator. The summation in equation (1) is performed over the electrons in whose electromagnetic field the test electron is located.

The friction and diffusion coefficients represent the ensemble-averaged rate of change of the momentum $\langle\Delta\boldsymbol{p}\rangle$ and of the product of momenta $\langle\Delta p_i \Delta p_j\rangle$, respectively.

The deviation of dynamic variables from the equilibrium values $\Delta x = (\Delta\boldsymbol{r}, \Delta\boldsymbol{p})$ using Eq. (1) can be writing in the form $\Delta\boldsymbol{x} = \sum_s \Delta\boldsymbol{x}_s$: $\Delta\boldsymbol{p}_s(t) = \int_{t_0}^{t} d\tau\, \mathbf{F}^{(s)}(\tau)$, $\Delta\boldsymbol{r}_s(t) = \int_{t_0}^{t} d\tau\,(t-\tau)\mathbf{W}^{(s)}(\tau)$, where

$$\boldsymbol{W}^{(s)}(\tau) = \frac{1}{m\gamma}\left[\mathbf{F}^{(s)}(\tau) - e\boldsymbol{\beta}\left(\boldsymbol{\beta}\mathbf{E}_s(\tau)\right)\right]. \qquad (3)$$

2. FRICTION COEFFICIENT

We obtain the expression for the radiation friction coefficient by expanding the microscopic force in a series in terms of small deviations of the electron's trajectory and momentum from their equilibrium values.

$$F_{Rf} = \langle\frac{d}{dt}p_z\rangle = \langle\Delta\mathbf{x}\frac{\partial}{\partial\mathbf{x}}F_z[x(t)]\rangle_{x=x^{(0)}(t)}. \qquad (4)$$

Substituting the expressions (2), (3) into the right-hand side of Eq. (4) and averaging over the initial values of the dynamic variables in the six-dimensional phase space of coordinates and momenta of electrons [3], we obtain

$$F_{Rf} = \int_{t_0}^{t} dt_1 \int_{\Omega_\mathrm{q}} \left\{(t-t_1)\mathbf{W}^{(s)}[X(t_1,q_{0s})]\frac{\partial}{\partial\boldsymbol{r}} + \right.$$
$$\left. + \boldsymbol{F}^{(s)}[X(t_1,q_{0s})]\frac{\partial}{\partial\boldsymbol{p}}\right\} \times$$
$$F_z^{(s)}[X(t,q_{0s})] f(q_{0s})\mathrm{v_z}(t_{0s})dq_{0s}\,, \qquad (5)$$

where $f(q_{0s})$ is the electron distribution function at $z = 0$; $q_{0s} = (x_{0s}, y_{0s}, t_{0s}, \boldsymbol{p}_{0s},)$ are the coordinates and momentum at the time t_{0s}; $X(t_1, q_{0s}) = \left\{x^{(0)}(t_1), t_1; x_s^{(0)}(t, q_{0s})\right\}$, Ω_q is the region of integration over the initial coordinates of the electrons.

The initial electron distribution function we write in the form

$$f(\boldsymbol{p}_{0s}) = n_b\delta(p_{x0s} + Kmc)\delta\left(p_{y0s}\right)w(p_{z0s}), \qquad (6)$$

where n_b is the average number density of electrons, $K = |e|H_0/mc^2k_u$, $\delta(x)$ is the Dirac delta function, $w(p_z)$ is an arbitrary function normalized to unity.

In the Eq. (5) the integration region Ω_q is determined by the relation:

$$c(t_1 - t_{0s}) \geq |\boldsymbol{r}_t(t_1) - \boldsymbol{r}_{0s}|, \qquad (7)$$

where $\boldsymbol{r}_t$ is the coordinate of the test particle.

Let us assume that the electron beam is a uniform cylindrical beam of radius r_b. Then the transverse coordinates of the electrons satisfy the condition

$$\sqrt{x_{0s}^2 + y_{0s}^2} \leq r_b. \qquad (8)$$

We express the longitudinal momentum distribution function in the beam as follows:

$$w(p_{z0s}) = \frac{1}{\sqrt{2\pi}p_{th}}\exp\left[-\frac{(p_{z0s}-p_{zm})^2}{2p_{\mathrm{th}}^2}\right], \qquad (9)$$

where p_{zm} and p_{th} are the mean value and initial thermal spread of the longitudinal momentum of electrons, respectively.

The pair interaction force of electrons $\boldsymbol{F}^{(s)}$ and the function $\boldsymbol{W}^{(s)}$ can be computed explicitly in the approximation $K \ll 1$. In addition, we will use the following assumptions. Let us find the friction coefficient for the electrons moving near the beam axis. The electromagnetic field of the emitter electrons moving behind the considered (test) electron is taken into account. The distance between two electrons, the pair interaction of which we are considering, is substantially greater than the thermal separation of these electrons relative to each other. The beam radius is larger than the wavelength of undulator radiation in the transverse direction.

Expanding the expressions for the electric and magnetic fields in powers of K and using (6), (9) the expression for the friction coefficient can be obtained:

$$F_{Rf} = -\frac{\pi e^4 k_u^2 n_b K^4 \gamma_m}{4mc^2} \int_0^z dz_1 \int_0^{\pi/2} d\theta \sin\theta \times \chi^3(\theta) r'_{max}(z_1,\theta) \times \{k_u(z-z_1)(\sin^2\theta + 2\cos\theta)\alpha_s[(z-z_1)\chi(\theta)] + \chi(\theta)\alpha[(z-z_1)\chi(\theta)]\}, \quad (10)$$

where $\alpha_s(x) = \exp\left(-\frac{x^2}{z_c^2}\right)\sin(\eta x)$, $\alpha(x) = \exp\left(-\frac{x^2}{z_c^2}\right)\cos(\eta x)$, $z_c = \frac{\sqrt{2}p_{zm}}{k_u p_{th}}$, $\eta = k_u\left(1-\frac{p_{zm}}{p_z}\right)$, $\chi = 1+\beta_{zs}\cos\theta$, $\gamma_{zs} = (1-\beta_{zs}^2)^{-1/2} = \gamma_{0s}/\sqrt{1+K^2}$.

Using relations (7) and (8), the expression for $r'_{max}(z_1,\theta)$ can be written in the following form

$$r'_{max}(z_1,\theta) = \min\left[\frac{z_1}{\gamma_{zm}^2(\beta_{zm}+\cos\theta)}, \frac{r_b}{\gamma_{zm}\sin\theta}\right]. \quad (11)$$

Let us first consider the case of pre-Brownian motion of electrons, when $z \ll z_c$. Neglecting the thermal spread of electron momenta, we expand the integrand (19) in a series in the small parameter $\eta z'\chi$ to quadratic terms. For $z < z_r$, we get

$$F_{Rf} = -A_{pB} z^2 \left[\frac{15}{16} + \frac{17}{60}\left(\frac{p_z}{p_{zm}} - 1\right) k_u^2 z^2\right], \quad (12)$$

where $A_{pB} = \frac{\pi e^4 k_u^2 n_b K^4}{2mc^2\gamma}$, $z_r = \beta_{zm}\gamma_{zm} r_b$.

For $z > z_r$ calculating the integral on the right-hand side of the Eq. (10), we obtain the following expression for the friction coefficient

$$F_{Rf} = -A_{pB} z_r z \left[5B(\zeta) + \left(\frac{p_z}{p_{zm}} - 1\right) k_u^2 z^2 B_\eta(\zeta)\right], \quad (13)$$

where $B_\eta(x) = 2.75(\mathrm{arctg}x - \pi/4)$, $\zeta = z/z_r$, and $B(x) = (7/8)(\mathrm{arctg}x - \pi/4) + (2/3)(1-1/x)$.

Let us now consider the solutions of Eq. (10) corresponding to the kinetic stage at $z \gg z_c$, assuming that $p_{th} \ll p_{zm}$.

For $z < z_r$ we find:

$$F_{Rf} = -A_{kin}(5z/3) x_p e^{-x_p^2}, \quad (14)$$

where $A_{kin} = \frac{\pi^{3/2} e^4 k_u n_b K^4 p_{zm}}{4cp_{th}^2}$ and $x_p = \frac{p_z - p_{zm}}{\sqrt{2}p_{th}}$.

For $z - z_r \gg z_c$ we obtain

$$F_{Rf} = -3A_{kin} z_r \left(arctg\zeta - \frac{\pi}{4}\right) x_p e^{-x_p^2}. \quad (15)$$

Eq. (10) was solved numerically for the following parameter values: $E_b = mc^2\gamma_0 = 17$ GeV, $r_b = 50$ μm, $H_0 = 1.4$ kOe, and $\lambda_u = 4$ cm.

Fig. 1 shows the dependences of the normalized value of the longitudinal friction coefficient $F_{nor} = F_{Rf}(K^2|F_{zR}|\, I_b/I_A)^{-1}$, on the normalized longitudinal momentum $(p_z - p_{zm})/p_{th}$ at the distance from the undulator entrance z=30 m (see Fig. 1,a), 60 m (see Fig. 1,b), $F_{zR} = -(2/3)r_e^2 H_0^2 \gamma^2 \beta_z^3$ is the radiation reaction force acting on an individual electron due to its own electromagnetic field, $r_e = e^2/mc^2$. In this calculation, we utilize $I_b = \pi|e|v_{z0} r_b^2 n_b$ and $I_A = mc^3/|e| = 17$ kA.

As can be seen from Fig. 1, radiation friction will increase the longitudinal momentum of electrons if it is less than the average value. At the same time, electrons with a longitudinal momentum greater than the average value of the momentum are slowed down. This may result reduction in the momentum spread in the beam. Since a change in momentum leads to a change in energy ($\Delta\gamma = \Delta p_z/mc$), the energy spread of electrons can also decrease.

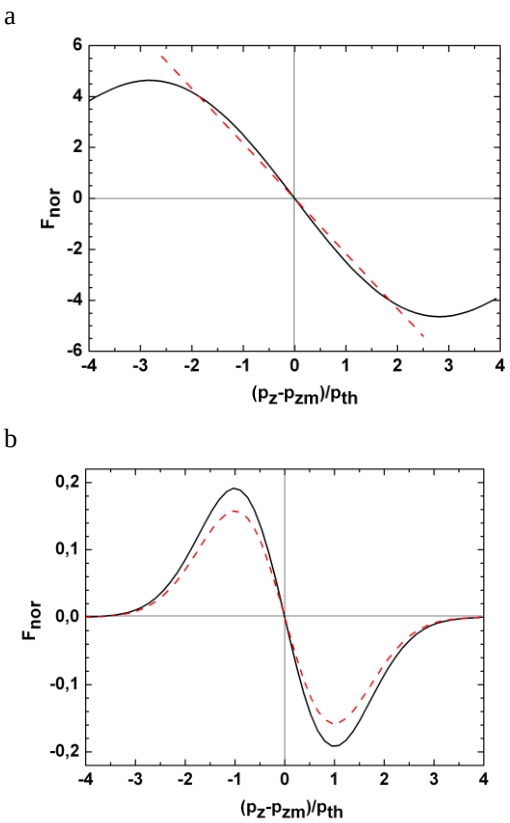


Fig. 1. Dependence of the normalized friction coefficient on the normalized longitudinal momentum. The black solid curves are the numerical solutions. The red dotted curves are given by the Eq. (13) (a) and Eq. (15) (b). p_{th}/p_{zm}=10^{-4} (a), 10^{-3} (b)

Fig. 2 shows the dependence of the normalized friction coefficient F_{nor} on the longitudinal coordinate z. The value F_{Rf} was calculated for the longitudinal momentum $p_z = p_{zm} - p_{th}$. The initial values of the longitudinal momentum spread correspond to the values z_c = 9 m (see Fig. 2,a) and 90 m (see Fig. 2,b).

As is seen from Fig. 2, at the pre-Brownian stage ($z < z_c/3$)) the values of the friction coefficient obtained both analytically and numerically coincide quite well. At the kinetic stage ($z > z_c$, Fig. 2,a) the values of the friction

coefficient obtained numerically for $z > 2z_c$, are greater than the values obtained analytically by approximately the value of friction coefficient at the stage preceding the kinetic stage of the system's evolution $F_{Rf}(z_c/2) \approx 0.03$.

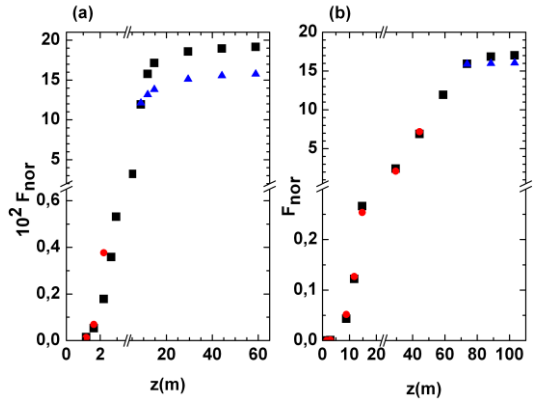


Fig. 2. Dependence of the normalized friction coefficient at $p_z = p_{zm} - p_{th}$ on the z coordinate. Black squares are numerical calculations. Analytical solutions are represented by red circles (Eq. (13)) and blue triangles (Eq. (15)). p_{th}/p_{zm}=10^{-3} (a), 10^{-4} (b)

It can be seen that at the pre-Brownian stage, the amplitude of the friction coefficient increases with increasing longitudinal coordinate z. At the kinetic stage, when $z - z_r \gg z_c$, the maximum value of the friction coeficient is practically independent of the z coordinate.

3. DIFFUSION COEFFICIENT

The expression for the diffusion coefficient obtained on the basis of the dynamics of individual particles motion can be written in the form [3]

$$D_z = \frac{d}{2dt}\left\langle (\Delta p_z)^2 \right\rangle = \int_{t_0}^{t} dt_1 \int_{\Omega_q} F_z^{(s)}\left[X(t,q_{0s})\right]\times \quad (16)$$

$$\times F_z^{(s)}\left[X(t_1,q_{0s})\right] f(q_{0s})\,\mathrm{v}_z(t_{0s})\,dq_{0s} \quad .$$

In the approximation of a small value of the undulator parameter, the expression for the diffusion coefficient is

$$D_z = \left(\frac{e^2\,\mathrm{K}^2 k_u}{\gamma}\right)^2 \int_{t_0}^{t} dt_1 \int_{-\infty}^{\infty} dp_{z0s}\gamma_{0s}^2 w(p_{z0s})\Phi\,, \quad (17)$$

where

$$\Phi = \iiint_{\Omega_q} \frac{G(\mathbf{r},t;q_{0s})G(\mathbf{r}_1,t_1;q_{0s})}{R_*(\mathbf{r},t;q_{0s})R_*(\mathbf{r}_1,t_1;q_{0s})} n_b\,\mathrm{v}_{zs}\,dx_{0s}dy_{0s}dt_{0s}\,. \quad (18)$$

Using the same assumptions as those used to derive the friction coefficient, we obtain

$$D_z = \frac{\pi e^4\,\mathrm{K}^4 k_u^2 n_b}{4\gamma^2\,\mathrm{v}_z}\int_0^z dz_1 \int_{-\infty}^{\infty} dp_{z0s}\gamma_{0s}^4 w(p_{z0s})\times \quad (19)$$

$$\times \int_0^{\pi/2} d\theta \sin\theta \chi^4(\theta)\cos\left[\gamma_{zs}^2 k_u \xi\chi(\theta)\right]\cdot r'_{\max}(z_1,\theta).$$

Substituting (9) into (19) and integrating over the longitudinal momentum using the value of the integral [10], we obtain the following expression for the diffusion coefficient

$$D_z(p_z) = \frac{\pi e^4\,\mathrm{K}^4 k_u^2 \gamma_m^4 n_b}{4\gamma^2\,\mathrm{v}_z}\int_0^z dz_1$$

$$\int_0^{\pi/2} d\theta \sin\theta\,\chi^4(\theta) r'_{\max}(z_1,\theta)\alpha[\chi(\theta)(z-z_1)]. \quad (20)$$

In the case of pre-Brownian motion of electrons, when $z << z_c$, from expression (20), we obtain

$$D_z = \frac{\pi e^4\,\mathrm{K}^4\,\gamma_m^2 k_u^2 n_b}{2\gamma^2\,\mathrm{v}_z^2} z\begin{cases}(15/16)z, & z < z_r\\ 2z_r B(z/z_r), & z > z_r\end{cases}, \quad (21)$$

where

$$B(x) = \frac{5}{3}\left(1-\frac{1}{x}\right) + \frac{35}{16}\left(arctgx - \frac{\pi}{4}\right) - \frac{2}{x}\ln\left(\frac{1+x^2}{2}\right) + \frac{15}{32x}\,.$$

In the case $z >> z_c$, of the kinetic stage, the asymptotic expression for the diffusion coefficient (20) at $z > z_r$ takes the form

$$D_z(p_z,z) = \frac{5\pi^{5/2}}{16}\left(\frac{e^2\,\mathrm{K}^2\,\gamma_m}{\gamma}\right)^2 \frac{n_b}{\mathrm{v}_z} k_u^2 z_r z_c \times$$

$$\times\left[B_2\left(\frac{z}{z_r}\right)e^{-\frac{(p_z-p_{zm})^2}{2p_{th}^2}} - \frac{8z_r z_c}{5\pi^{3/2}z^2}\phi\left(\frac{p_z-p_{zm}}{\sqrt{2}\,p_{th}}\right)\right], \quad (22)$$

where $B_2(x) = \frac{2}{\pi}arctgx - \frac{1}{2} + \frac{22}{15\pi} - \frac{6}{5\pi x}$,

$\phi(x) = 1 - 2xe^{-x^2}\int_0^x e^{t^2}dt$.

Expression (22) at $z >> z_r$ takes the form

$$D_z = \frac{5\pi^{5/2}e^4\,\mathrm{K}^4\,\gamma_m^2 n_b}{16\gamma^2\,\mathrm{v}_z} k_u^2 z_r z_c e^{-\frac{(p_z-p_{zm})^2}{2p_{th}^2}}\,. \quad (23)$$

The diffusion coefficient is independent of z, that corresponds to the kinetic stage of the evolution of a system consisting of a large number of particles (see, for example, [7-9]).

4. DISCUSSION

The presented analysis shows that in a flow of electrons moving in a periodic field the interaction of individual particles with each other through the electromagnetic fields they create leads to the effects of radiation friction and diffusion in momentum space.

The force act on electron is proportional to the number of electrons emitter in whose electromagnetic field it is located. Therefore, the friction coefficient is proportional to the electron beam density n_b.

For the distances $0 \le z \ll z_c$, the increase in the friction coefficient with increasing z coordinate is due, on the one hand, to the fact that the force of pair interaction of electrons is practically independent of the z coordinate, and on the other hand, to the increase in the number of electrons emitter in whose electromagnetic field the test electron is located.

At the kinetic stage $z \ge z_c$, the amplitude of the friction coefficient takes on its greatest value at $|p_z - p_{zm}| = p_{th}$. The dependence of the friction force on the momentum in the electromagnetic field of undulator radiation is similar to the dependence of the

friction force on the velocity in Coulomb collisions of charged particles in the electron cooling method [11].

From Eqs. (13), (15), and Fig. 2 it follows that the maximum value of the friction coefficient can be greater than the radiation reaction force acting on an individual electron. For example, for the undulator and beam parameters cited above, and beam current $I_b = 5$ kA, and initial energy spread $\Delta\gamma/\gamma = 10^{-4}$, the value of the radiation friction coefficient at $p_z = p_{zm} - p_{th}$ will be greater than the radiation reaction force at distances greater than 42 m from the entrance to the undulator.

If we use an electron beam with an ultrahigh peak current of 0.1 MA [12], an energy of 10 GeV, a radius of 20 μm and the same values of the energy spread and undulator parameters, the value of the radiation friction coefficient for the same values of the longitudinal momentum of the electrons will be greater than the radiation reaction force at distances greater than 9 m from the entrance to the undulator.

Study of the evolution of the beam electron distribution in momentum space is of interest to the development of X-ray FELs, based on the self-amplification of spontaneous emission by an ultrarelativistic electron beam passing through an undulator. Radiation friction and diffusion [4] may lead to a change in the initial distribution of electron momenta, and consequently an effect on the formation of stimulated emission.

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КОЕФІЦІЄНТИ ТЕРТЯ І ДИФУЗІЇ В РЕЛЯТИВІСТСЬКОМУ ЕЛЕКТРОННОМУ ПУЧКУ, ЩО РУХАЄТЬСЯ В ОНДУЛЯТОРІ

В.В. Огнівенко

Розглянуто вплив некогерентного поля спонтанного випромінювання релятивістських електронів, що рухаються в просторово періодичному магнітному полі ондулятора, на зміну середнього і середньоквадратичного поздовжнього імпульсу електронів. Отримані коефіцієнти тертя і дифузії й визначені залежності їх як від відстані, вздовж ондулятора, так і від величини початкового енергетичного розкиду електронів.